Math 2120

4/13/20

Week 12

11.4 continued...

Recall

$\vec{v} = \langle a, b, c \rangle, \ \vec{w} = \langle d, e, f \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

3x3 determinant

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= \vec{i} \begin{vmatrix} b & c \\ e & f \end{vmatrix} - \vec{j} \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \vec{k} \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} \qquad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} \qquad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

2×2 determinants

$$\begin{vmatrix} x & y \\ w & z \end{vmatrix} = xz - yw$$

$$\begin{vmatrix} x & y \\ w & z \end{vmatrix} \qquad \begin{vmatrix} x & y \\ w & z \end{vmatrix}$$

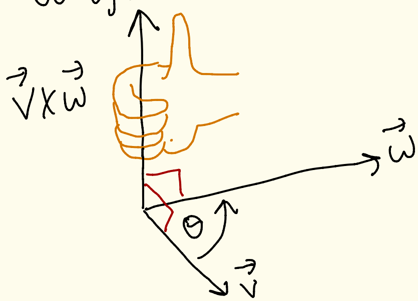
We usually only cross product non-zero vectors.

Properties of cross product

Let $\vec{v}$ and $\vec{w}$ be nonzero vectors in 3-dimensions [ie $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$]

① $\vec{v} \times \vec{w}$ is orthogonal/perpendicular to both $\vec{v}$ and $\vec{w}$.

② The direction of $\vec{v} \times \vec{w}$ is given by the right hand rule. That is, if the fingers of your right hand curl in the direction of a rotation (through the angle θ less than $180°$) from $\vec{v}$ to $\vec{w}$

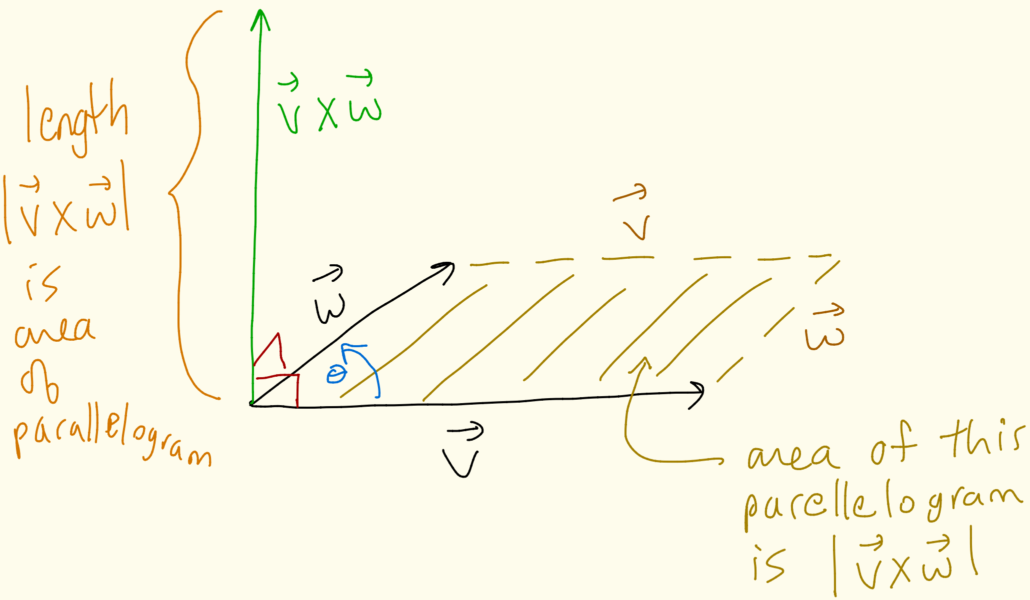


then your thumb points in the direction of $\vec{v} \times \vec{w}$.

(3) If θ is the angle between $\vec{v}$ and $\vec{w}$ $(0 \le \theta \le \pi)$ then

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\theta)$$

Moreover, this is the area of the parrallelogram with sides $\vec{v}$ and $\vec{w}$.



4. Let $\vec{u}$ also be a non-zero vector. Let α, β be scalars, Then:

a. $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$

b. $(\alpha \vec{v}) \times (\beta \vec{w}) = \alpha\beta(\vec{v} \times \vec{w})$

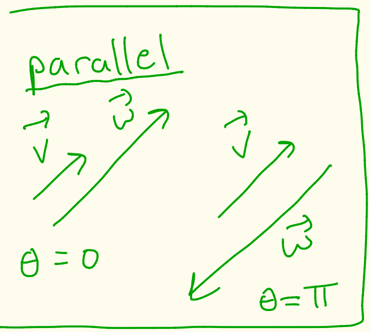
c. $\alpha(\vec{v} \times \vec{w}) = (\alpha \vec{v}) \times \vec{w}$
$= \vec{v} \times (\alpha \vec{w})$

d. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

e. $(\vec{u} + \vec{v}) \times \vec{w} = \underbrace{\vec{u} \times \vec{w} + \vec{v} \times \vec{w}}$

same as $-(\vec{w} \times \vec{u}) - \vec{w} \times \vec{v}$ by property (a)

Question asked in class



5. $\vec{v}$ and $\vec{w}$ are parallel (ie $\theta = 0$ or $\theta = \pi$) if and only if $\vec{v} \times \vec{w} = \vec{0}$

Pg 5

Ex: $\boxed{\begin{aligned}\vec{v} &= \langle 0, 2, 2 \rangle \\ \vec{w} &= \langle 0, -2, 2 \rangle\end{aligned}}$

Recall:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{vmatrix} \qquad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{vmatrix} \qquad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{vmatrix}$$

$$= \vec{i}\left[(2)(2) - (2)(-2)\right] - \vec{j}\left[(0)(2) - (2)(0)\right] + \vec{k}\left[(0)(-2) - (2)(0)\right]$$

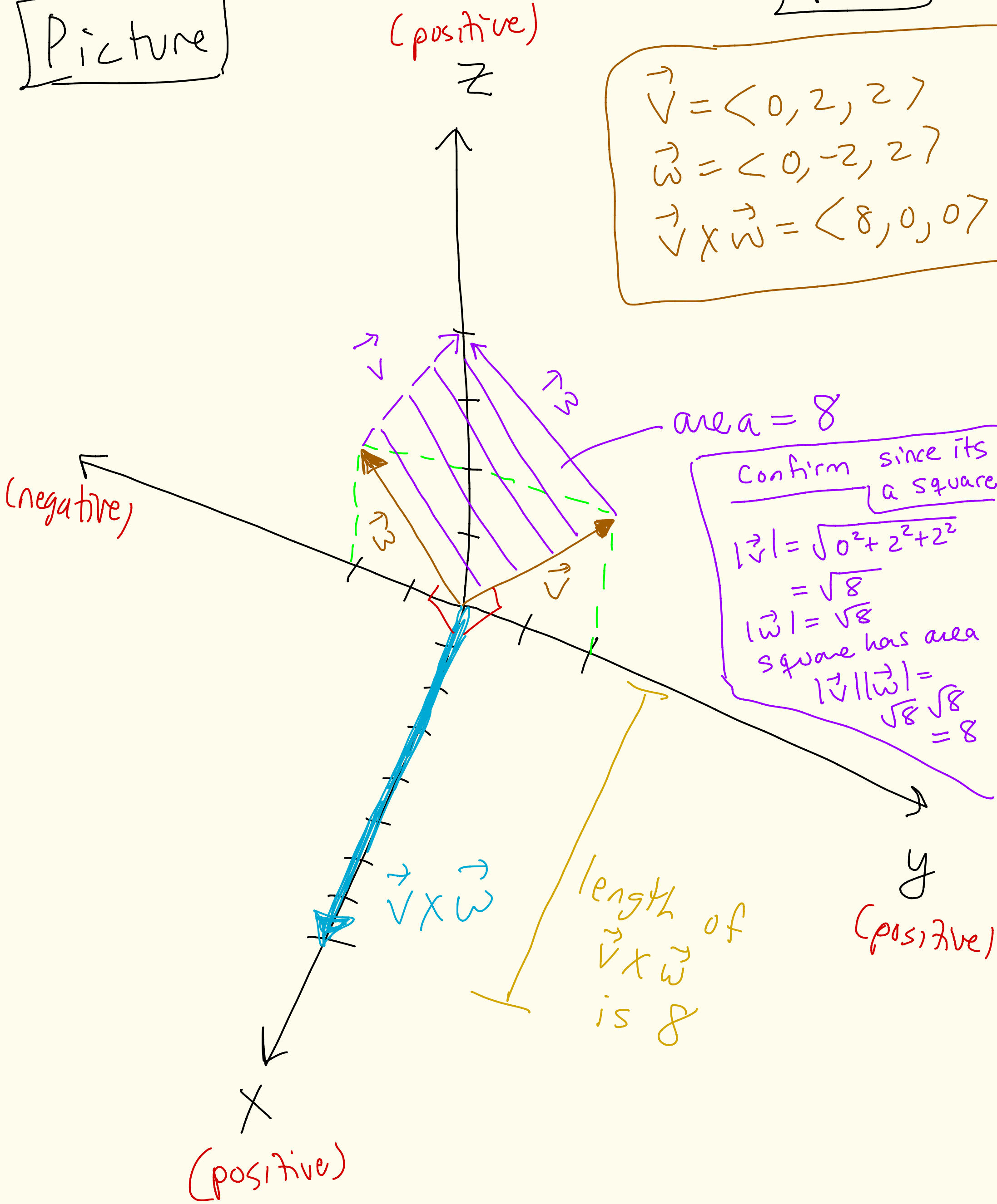
$$= 8\vec{i} + 0\vec{j} + 0\vec{k} = \langle 8, 0, 0 \rangle$$

Picture

$\vec{v} = \langle 0, 2, 2 \rangle$

$\vec{w} = \langle 0, -2, 2 \rangle$

$\vec{v} \times \vec{w} = \langle 8, 0, 0 \rangle$



Confirm since its a square

$|\vec{v}| = \sqrt{0^2 + 2^2 + 2^2}$
$= \sqrt{8}$

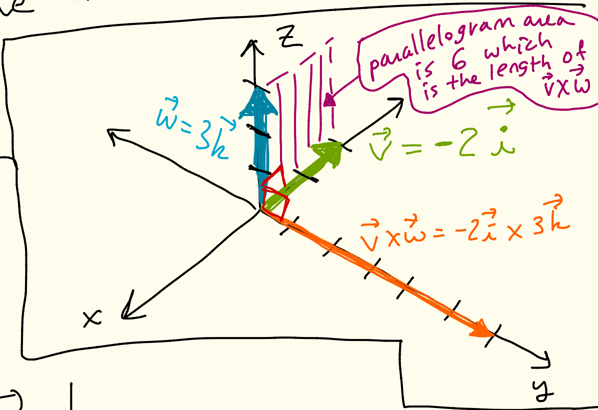
$|\vec{w}| = \sqrt{8}$

square has area
$|\vec{v}||\vec{w}| =$
$\sqrt{8}\sqrt{8}$
$= 8$

Calculus workshop

11.4

(19) Calculate $(-2\vec{i}) \times (3\vec{k})$ and draw the answer with the vectors.



$\vec{v} = -2\vec{i} = \langle -2, 0, 0 \rangle$

$\vec{w} = 3\vec{k} = \langle 0, 0, 3 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} -2 & 0 \\ 0 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= \vec{i}(0 \cdot 3 - 0 \cdot 0) - \vec{j}((-2) \cdot 3 - 0 \cdot 0) + \vec{k}((-2) \cdot 0 - 0 \cdot 0)$$

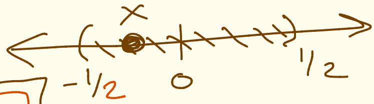
$6\vec{j} = \langle 0, 6, 0 \rangle$

14 $f(x) = (1+2x)^{-1}$

Find the Maclaurin series (centered at $a=0$) and its interval of convergence.

Answer: $\displaystyle\sum_{k=0}^{\infty} (-2)^k x^k$

interval of conv. $\left(-\frac{1}{2}, \frac{1}{2}\right)$



Method #1:

$$\frac{1}{1+2x} = \frac{1}{1-(-2x)}$$

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots$$

when $|r| < 1$ or $-1 < r < 1$

works when:

$|-2x| < 1$

$|-2||x| < 1$

$|x| < \frac{1}{2}$

$-\frac{1}{2} < x < \frac{1}{2}$

$$\sum_{k=0}^{\infty} (-2x)^k = \sum_{k=0}^{\infty} (-1)^k 2^k x^k = \boxed{\sum_{k=0}^{\infty} (-2)^k x^k}$$

Method 2:

Maclaurin series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

$f(x) = (1+2x)^{-1} \leftarrow f^{(0)}(0) = 1$

$f'(x) = -(1+2x)^{-2} \cdot 2 = -2(1+2x)^{-2}$

$f^{(1)}(0) = -2$

$f''(x) = (-2)(-2) \cdot (1+2x)^{-3} \cdot 2$

$= 2^3 (1+2x)^{-3} \leftarrow f^{(2)}(0) = 2^3$

$f'''(x) = 2^3(-3) \cdot (1+2x)^{-4} \cdot 2$

$= 2^4(-3) \cdot (1+2x)^{-4}$

$f^{(3)}(0) = 2^4(-3)$

Maclauren

$\frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$

$\frac{1}{1} \cdot 1 + \frac{(-2)}{1}x + \frac{2^3}{2}x^2 + \frac{2^4(-3)}{3 \cdot 2 \cdot 1}x^3 + \cdots = 1 - 2x + 2^2x^2 - 2^3x^3 + \cdots$

$\boxed{\text{On test}}$

$$= \cdots = \frac{5}{\infty} = 0$$

$$= \cdots = 0$$

"$\frac{5}{\infty}$"

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telescoping series #3

see solutions ← see extra problem solutions

Question from class: