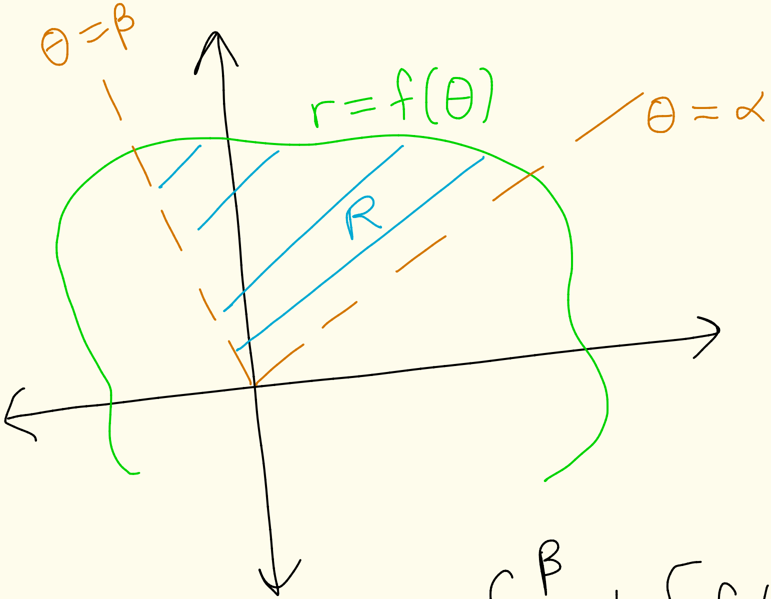
Math 2120

4/22/20

From last time



$$\text{area of } R = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta$$

make sure:

$$\boxed{0 < \beta - \alpha \le 2\pi}$$

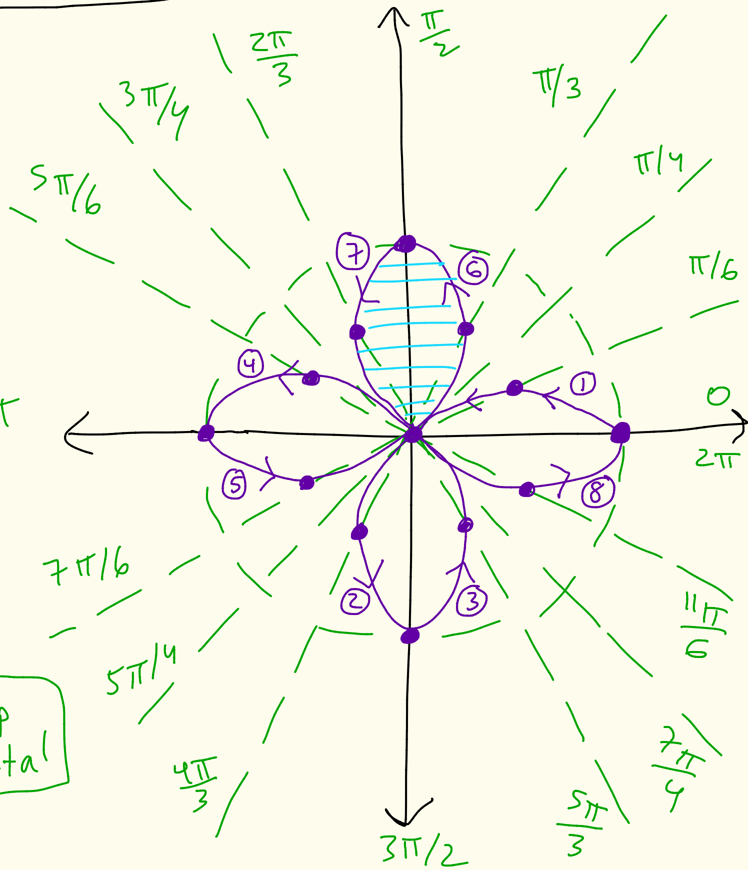
$\beta > \alpha$

β & α aren't more than 2π apart

Ex: Find the area of one petal in the 4-leafed rose given by $r=\cos(2\theta)$

θ	$r=\cos(2\theta)$
0	1
$\pi/6$	$1/2$
$\pi/4$	0
$\pi/3$	$-1/2$
$\pi/2$	-1
$2\pi/3$	$-1/2$
$3\pi/4$	0
$5\pi/6$	$1/2$
π	1
$7\pi/6$	$1/2$
$5\pi/4$	0
$4\pi/3$	$-1/2$
$3\pi/2$	-1
$5\pi/3$	$-1/2$
$7\pi/4$	0
$11\pi/6$	$1/2$
2π	1

top petal



area of top petal $= \displaystyle\int_{5\pi/4}^{7\pi/4} \frac{1}{2}\left[\cos(2\theta)\right]^2 d\theta =$

Pg 3

$$= \int_{5\pi/4}^{7\pi/4} \frac{1}{2}\cos^2(2\theta)\, d\theta$$

$$= \int_{5\pi/4}^{7\pi/4} \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}\cos(4\theta)\right] d\theta$$

$$\boxed{\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)}$$

$$= \int_{5/4}^{7\pi/4} \left[\frac{1}{4} + \frac{1}{4}\cos(4\theta)\right] d\theta$$

$$= \left[\frac{1}{4}\theta + \frac{1}{4}\left[\frac{1}{4}\sin(4\theta)\right]\right]_{5\pi/4}^{7\pi/4}$$

$$= \frac{1}{4}\left(\frac{7\pi}{4}\right) + \frac{1}{16}\sin\left(4\cdot\frac{7\pi}{4}\right) - \left[\frac{1}{4}\left(\frac{5\pi}{4}\right) + \frac{1}{16}\sin\left(4\cdot\frac{5\pi}{4}\right)\right]$$

$\sin(7\pi) = 0$

$\sin(5\pi) = 0$

$$= \frac{1}{4}\left(\frac{7\pi}{4}\right) - \frac{1}{4}\left(\frac{5\pi}{4}\right) = \frac{1}{4}\left[\frac{2\pi}{4}\right] = \frac{\pi}{8} \approx 0.3926\ldots$$

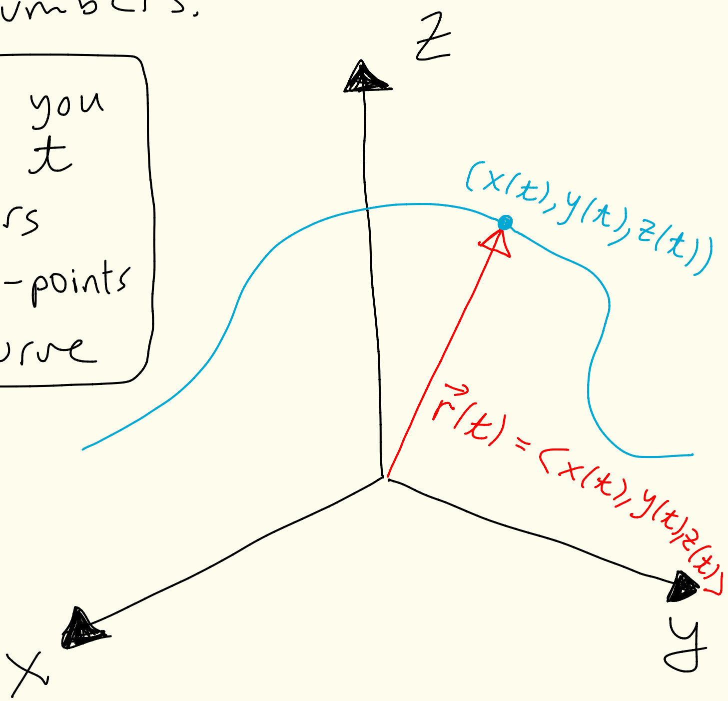
11.5 – Lines and Curves in Space

A <u>vector-valued function</u> in 3d is a function of the form

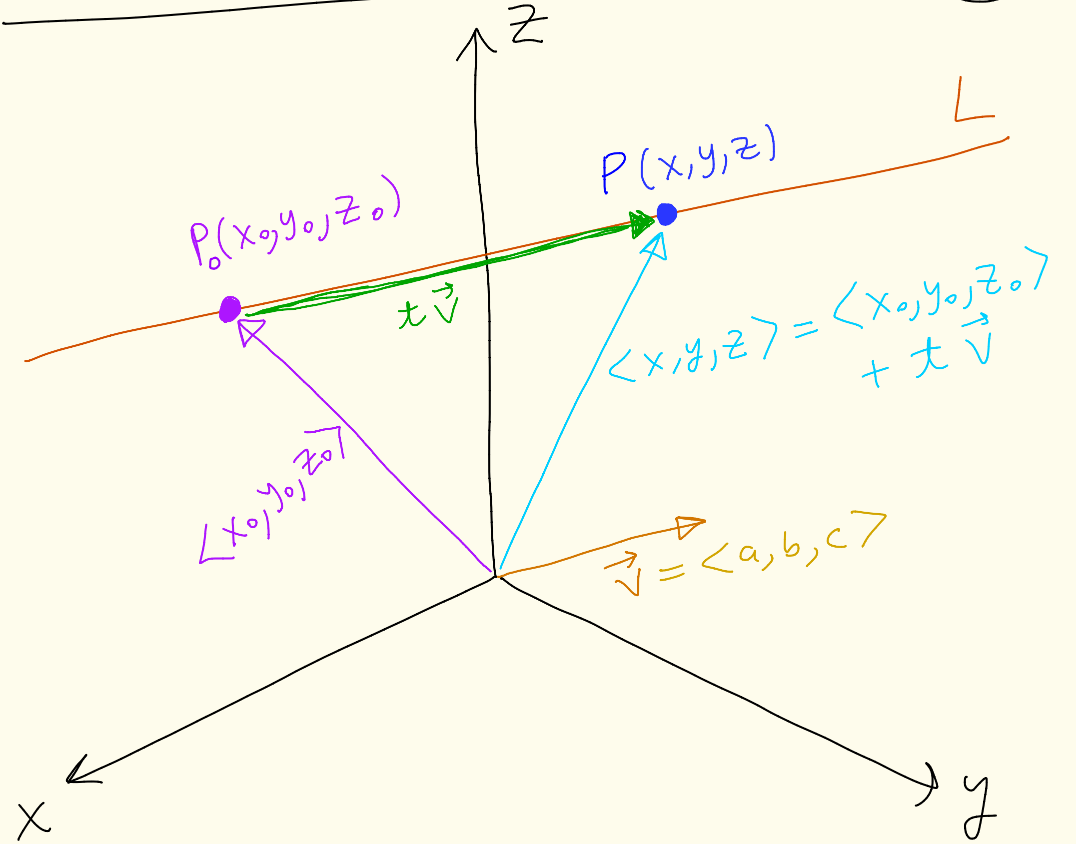
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

where t, $x(t)$, $y(t)$, $z(t)$ are real numbers.

Idea is: As you plug in various t you get vectors and their end-points trace out a curve in 3d.



Equation of a line in 3d



Let L be a line in 3d. Let $P_0(x_0,y_0,z_0)$ be a point on the line. Let $\vec{v}=\langle a,b,c\rangle$ be a vector in the direction of L, ie let $\vec{v}$ be parallel to L. Let $P(x,y,z)$ be another point on L. Pick t such that $t\vec{v} = \overrightarrow{P_0P}$. Then $\langle x,y,z\rangle = \langle x_0,y_0,z_0\rangle + t\vec{v}$.

$$= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

$$= \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

$$= \langle \underbrace{x_0 + ta}_{x}, \underbrace{y_0 + tb}_{y}, \underbrace{z_0 + tc}_{z} \rangle$$

Eqn of a line in 3d

An equation of the line passing through (x_0, y_0, z_0) in the direction of the vector $\vec{v} = \langle a, b, c \rangle$ is given by

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\vec{v}$$

where t is any real number

or

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

where t is any real number

Workshop

pg 7

Chapter 9 stuff

9.2 –
- Find the interval of convergence of $\sum a_n(x-a)^n$ using the ratio test and testing endpoints
- Finding power series via integration and differentiation

9.3 –
- Finding Taylor/Maclaurin series either by 9.2 methods or explicitly calculating $f^{(k)}(a)$ and using the formula $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

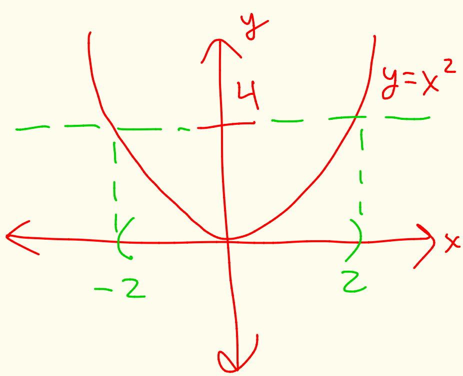
9.4 –
- Using the power series to solve limit problems
- differentiate power series (#25–32)
- Given a power series turn it into a function (HW#55, 57, 59)

57 Identify

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k} = \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^k}{4^k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{-x^2}{4}\right)^k = \frac{1}{1-\left(\frac{-x^2}{4}\right)} = \boxed{\frac{1}{1+\frac{x^2}{4}} \quad -2<x<2}$$



if $\left|\frac{-x^2}{4}\right| < 1$

$|x^2| < 4$

$$\boxed{\begin{array}{c} x^2<4 \\ -2<x<2 \end{array}}$$

9.3

17 Find the Maclaurin series for $f(x) = 3^x$ and its radius of convergence.

Note: We don't have a function to intergrate or differentiate like in the example

$$\ln(1+4x) = \frac{1}{4}\int \frac{1}{1+4x}\,dx + C$$

$$= \frac{1}{4}\int \left(1 - 4x + (-4x)^2 + \dots\right) dx$$

$$\frac{d}{dx} 3^x = (3^x)\ln(3)$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k$$

$$f^{(0)}(x) = 3^x$$

$$\frac{d}{dx} a^x = \ln(a) \cdot a^x$$

$$\frac{d}{dx} e^x = \ln(e) \cdot e^x = 1 \cdot e^x = e^x$$

Pg 10

$$f^{(1)}(x) = \ln(3) \cdot 3^x$$

$$f^{(2)}(x) = \ln(3) \cdot [\ln(3) \cdot 3^x] = [\ln(3)]^2 \cdot 3^x$$

$$f^{(3)}(x) = [\ln(3)]^2 \cdot \ln(3) \cdot 3^x = [\ln(3)]^3 \cdot 3^x$$

$$\vdots$$

$$f^{(k)}(x) = [\ln(3)]^k \cdot 3^x$$

$$f^{(k)}(0) = [\ln(3)]^k \cdot 3^0 = [\ln(3)]^k$$

Maclaurin series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{[\ln(3)]^k}{k!} x^k$$

Where does this converge?

$$\sum_{k=0}^{\infty} \frac{[\ln(3)]^k}{k!} x^k$$

$$L = \lim_{k\to\infty} \left| \frac{\dfrac{[\ln(3)]^{k+1} x^{k+1}}{(k+1)!}}{\dfrac{(\ln(3))^k x^k}{k!}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{\ln(3)^k} \cdot \ln(3) \cancel{x^k} x}{(k+1) \cdot \cancel{k!}} \cdot \frac{\cancel{k!}}{\cancel{(\ln(3))^k} \cdot \cancel{x^k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\ln(3)}{k+1} \right| |x|$$

$$= |x| \lim_{k\to\infty} \left| \frac{\ln(3)}{k+1} \right| = |x| \cdot 0 = 0$$

Since $0 \le L < 1$, the series converges for all x.

Ex: Find a power series for

$$f(x) = \frac{x}{(1+x^2)^2}.$$

$$\frac{d}{dx} \underbrace{\frac{1}{1+x^2}}_{(1+x^2)^{-1}} = -(1+x^2)^{-2} \cdot 2x$$

$$= -2\left[\frac{x}{(1+x^2)^2}\right]$$

So, $$\frac{x}{(1+x^2)^2} = -\frac{1}{2}\frac{d}{dx}\left[\frac{1}{1+x^2}\right]$$

$$= -\frac{1}{2}\frac{d}{dx}\left[\frac{1}{1-(-x^2)}\right] = -\frac{1}{2}\frac{d}{dx}\sum_{k=0}^{\infty}(-x^2)^k$$

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k$$

$|r| < 1$

$|-x^2| < 1$

$|x^2| < 1$

$-1 < x < 1$

$=$

$$= -\frac{1}{2}\frac{d}{dx}\sum_{k=0}^{\infty} \underbrace{(-1)^k x^{2k}}_{(-x^2)^k = (-1)^k (x^2)^k}$$

above $k=0$ is constant so change to $k=1$ below

$$= -\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k \cdot 2k\, x^{2k-1}$$

$-1<x<1$ but since differentiated endpoints might change

So, $\dfrac{x}{(1+x^2)^2} = -\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k \cdot 2k\, x^{2k-1}$

for $-1<x<1$. But it might also converge at the endpoints since we differentiated to find the series.

You can check, it won't converge at $x=1$ or $x=-1$.