Math 2120

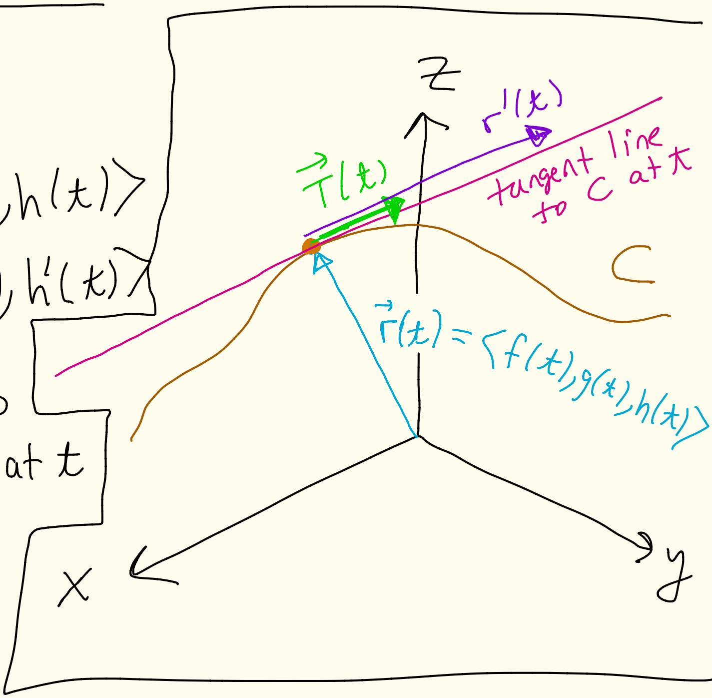
4/27/20

Last time

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

$\vec{r}\,'(t)$ is parallel to the tangent line at t of the curve C traced out by $\vec{r}(t)$.



You can also define the unit tangent vector at t by

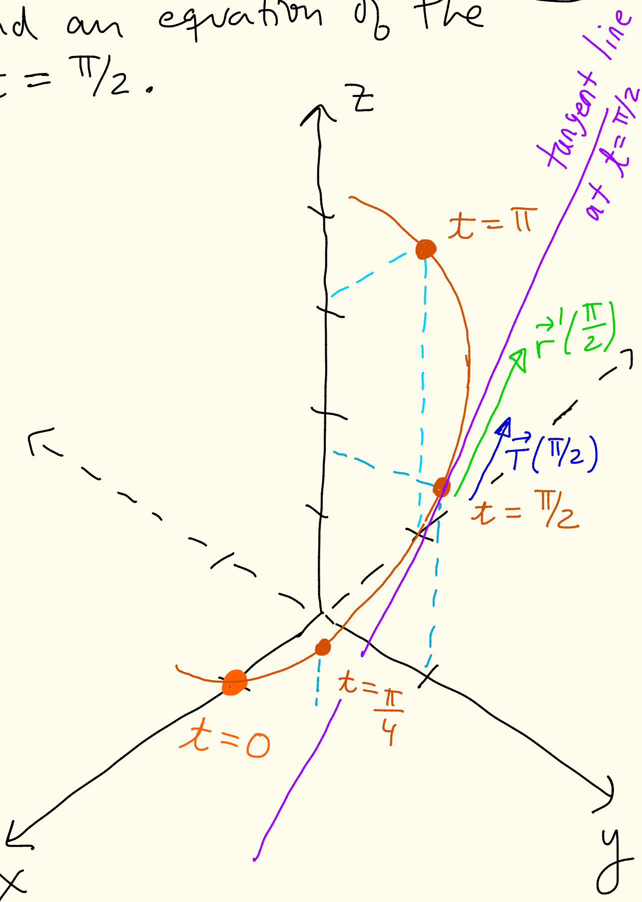
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad \text{provided } |\vec{r}\,'(t)| \neq 0$$

(remember: unit vector just means a vector of length 1)

Ex: Sketch $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

Find $\vec{r}\,'(t)$, $\vec{T}(t)$, and an equation of the tangent line at $t = \pi/2$.

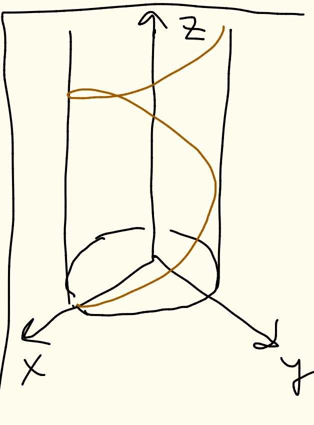
t	$\vec{r}(t)$
0	$\langle 1, 0, 0 \rangle$
$\pi/4$	$\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} \rangle$ $\approx \langle 0.7, 0.7, 0.79 \rangle$
$\pi/2$	$\langle 0, 1, \pi/2 \rangle$ $\approx \langle 0, 1, 1.57 \rangle$
$3\pi/4$	$\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{3\pi}{4} \rangle$ $\approx \langle -0.7, 0.7, 2.4 \rangle$
π	$\langle -1, 0, \pi \rangle$



$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$

$\vec{r}\,'(\pi/2) = \langle -1, 0, 1 \rangle$

$$\vec{T}(\pi/2) = \frac{\vec{r}\,'(\pi/2)}{|\vec{r}\,'(\pi/2)|} = \frac{\langle -1, 0, 1 \rangle}{\sqrt{(-1)^2 + 0^2 + 1^2}} = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$



Tangent line at $\langle 0, 1, \pi/2 \rangle$:

$x = 0 + (-1)t = -t$

$y = 1 + 0t = 1$

$z = \pi/2 + 1 \cdot t = \pi/2 + t$

- Look at what we said in workshop last week where I outlined Chapter 9.
- A few other things to definitely know:

— Memorize:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Converge for all x

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad (-1 < x < 1)$$

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}$$

$\vec{a}\cdot\vec{b}$

$\vec{a}\times\vec{b}$

$|\vec{a}|$

unit vectors: $\frac{\vec{a}}{|\vec{a}|}$

11.3

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(32) $\vec{u} = \langle 13, 0, 26 \rangle$

$\vec{v} = \langle 4, -1, -3 \rangle$

Find $\text{proj}_{\vec{v}}(\vec{u})$

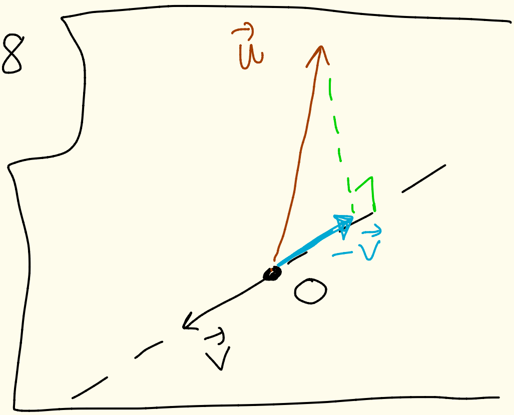
Side question:

Are $\vec{u}$ and $\vec{v}$ orthogonal?

No, because $\vec{u} \cdot \vec{v} \neq 0$

Recall: $\text{proj}_{\vec{v}}(\vec{u}) = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$$\vec{u} \cdot \vec{v} = (13)(4) + (0)(-1) + (26)(-3)$$
$$= 52 + 0 - 78$$
$$= -26$$



$$|\vec{v}| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26}$$
$$|\vec{v}|^2 = 26$$

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{-26}{26}\langle 4, -1, -3 \rangle = -\langle 4, -1, -3 \rangle$$
$$= \langle -4, 1, 3 \rangle$$
$$= -\vec{v}$$

9.3

$$\boxed{\frac{d}{dx} a^x = \ln(a) \cdot a^x}$$

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$f^{(0)}(x) = 10^x$

$f^{(1)}(x) = \ln(10) \cdot 10^x$

$f^{(2)}(x) = \ln(10) \cdot \ln(10) \cdot 10^x = [\ln(10)]^2 \cdot 10^x$

$f^{(3)}(x) = [\ln(10)]^2 \cdot \ln(10) \cdot 10^x = [\ln(10)]^3 \cdot 10^x$

$f^{(k)}(x) = [\ln(10)]^k \cdot 10^x$

Taylor series at $a = 2$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{[\ln(10)]^k \cdot 10^2}{k!}(x-2)^k$$

Ex: Find the Maclaurin series for $f(x) = \ln(2+x^4)$ and its interval of convergence. (pg 6)

Hint: $\dfrac{d}{du}\ln(2+u) = \dfrac{1}{2+u} = \ldots$

$\dfrac{d}{dx}\ln(2+x^4) = \dfrac{4x^3}{2+x^4} = \ldots$

$$\frac{4x^3}{2+x^4} = 4x^3\left[\frac{1}{2+x^4}\right]$$

$$= 4x^3 \cdot \frac{1}{2}\left[\frac{1}{1+\frac{x^4}{2}}\right]$$

$$= 2x^3\left[\frac{1}{1-\left(-\frac{x^4}{2}\right)}\right] = 2x^3\sum_{k=0}^{\infty}\left(-\frac{x^4}{2}\right)^k$$

$$\boxed{\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k, \quad |r|<1}$$

$\left|-\frac{x^4}{2}\right| < 1 \Rightarrow |x^4| < 2$

$-2^{1/4} < x < 2^{1/4}$

$$\frac{4x^3}{2+x^4} \underset{\uparrow \; -2^{1/4} < x < 2^{1/4}}{=} 2x^3 \sum_{k=0}^{\infty} \left(\frac{-x^4}{2}\right)^k = 2x^3 \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{2^k}$$

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$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k-1}} x^{4k+3}$$

$$\ln(2+x^4) = C + \int \frac{4x^3}{2+x^4}\,dx$$

$$= C + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k-1}} \frac{x^{4k+4}}{4k+4}$$

What is C?

$$\ln(2+0) = C + \underbrace{\Big[\underbrace{\frac{1}{2^{-1}} \frac{0^4}{4}}_{k=0} + \underbrace{\frac{-1}{2^0} \cdot \frac{0^8}{8}}_{k=1} + \dots\Big]}_{0}$$

$$C = \ln(2)$$

$$\ln(2+x^4) = \ln(2) + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k-1}} \frac{x^{4k+4}}{4k+4}$$

definitely converges $-2^{1/4} < x < 2^{1/4}$

It might also converge at endpoints

Evaluate using Taylor series

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$$\lim_{x\to 0} \frac{3\tan^{-1}(x) - 3x + x^3}{4x^5} \quad \text{" } = \frac{0}{0} \text{ "}$$

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2} + C = C + \int \frac{dx}{1-(-x^2)}$$

$$= C + \int \sum_{k=0}^{\infty} (-x^2)^k \, dx$$

$|-x^2| < 1$

$|x|^2 < 1$

$-1 < x < 1$

$$= C + \int \sum_{k=0}^{\infty} (-1)^k x^{2k} \, dx$$

$$= C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$-1 < x < 1$

might converge at endpoints

It actually converges on $-1 \le x \le 1$ if you check the endpoints

check for C

$0 = \tan^{-1}(0)$

$$= C + \underbrace{\left[\frac{0^1}{1} - \frac{0^3}{3} + \dots\right]}_{0}$$

$= C$

So, $C = 0$.

So,

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \quad \text{where } -1 \le x \le 1$$

$$\lim_{x\to 0} \frac{3\tan^{-1}(x) - 3x + x^3}{4x^5}$$

$$= \lim_{x\to 0} \frac{3\Big[\underbrace{x}_{k=0} - \underbrace{\frac{x^3}{3}}_{k=1} + \underbrace{\frac{x^5}{5}}_{k=2} - \underbrace{\frac{x^7}{7}}_{k=3} + \cdots\Big] - 3x + x^3}{4x^5}$$

$$= \lim_{x\to 0} \frac{\Big[\cancel{3x} - \cancel{x^3} + \frac{3}{5}x^5 - \frac{3}{7}x^7 + \cdots\Big] - \cancel{3x} + \cancel{x^3}}{4x^5}$$

$$= \lim_{x\to 0} \frac{3/5\, x^5 - 3/7\, x^7 + \cdots}{4x^5}$$

$\cdots$ has terms of x^9 or higher powers

$$= \lim_{x\to 0} \frac{(3/5)}{4} - \frac{(3/7)}{4} x^2 + \cdots = \frac{(3/5)}{4} + 0 = 3/20$$