Math 2120

4/28/20

No class tomorrow

Test 3

Final – Tuesday 12th (12–2)

11.8 – Length of curves

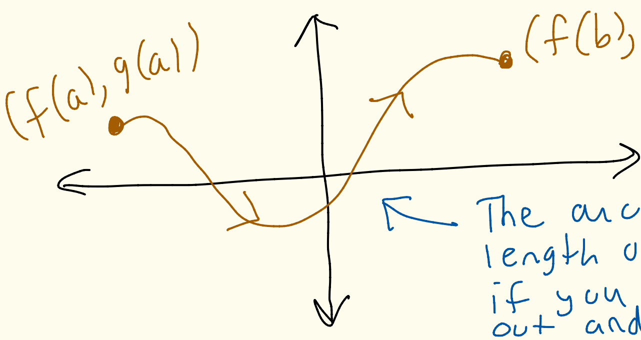
$\vec{r}\,'(t) = \langle f'(t), g'(t)\rangle$

$|\vec{r}\,'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2}$

2D version

Consider the curve given by $\vec{r}(t) = \langle f(t), g(t)\rangle$ where $f'(t)$ and $g'(t)$ exist and are continuous for $a \le t \le b$. The arc length of the curve between $(f(a), g(a))$ and $(f(b), g(b))$ is

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt = \int_a^b |\vec{r}\,'(t)|\,dt$$



The arclength is length of this curve if you flattened it out and measured it

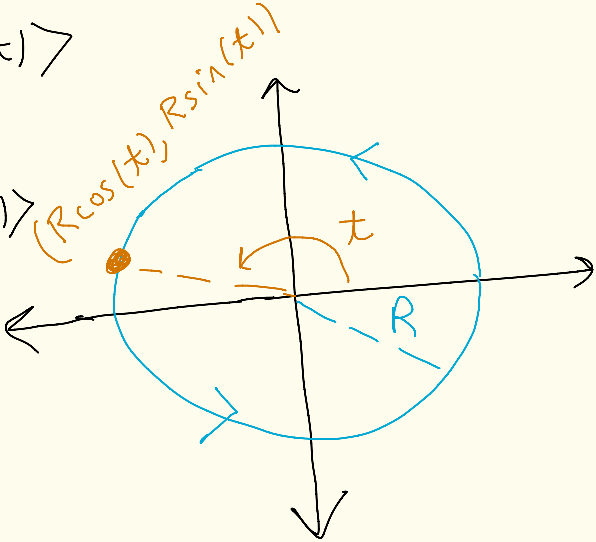
Ex: Find the circumference of a circle of radius R.

pg 2

$\vec{r}(t) = \langle R\cos(t), R\sin(t) \rangle$

$0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -R\sin(t), R\cos(t) \rangle$



circumference =

arclength of circle =

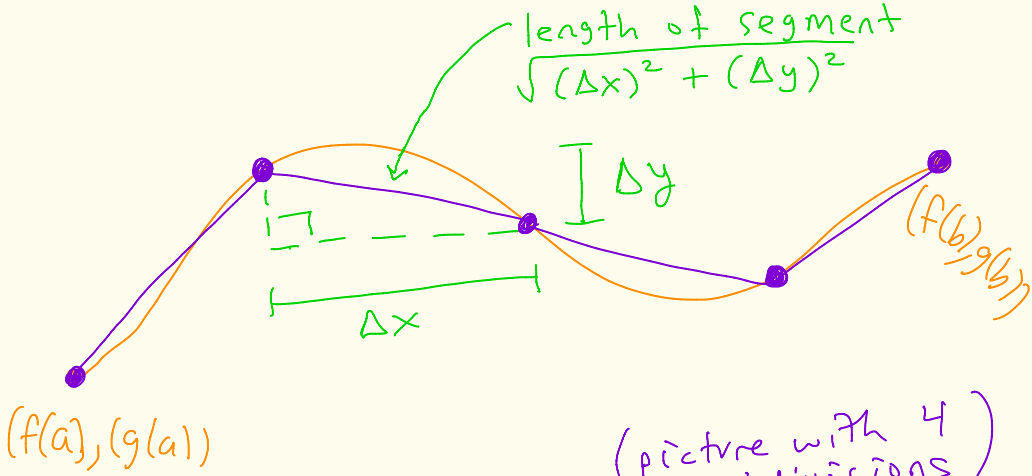
$$= \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} \sqrt{(-R\sin(t))^2 + (R\cos(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}\, dt$$

$$= \int_0^{2\pi} \underbrace{\sqrt{R^2}}_{R} \underbrace{\sqrt{\sin^2(t) + \cos^2(t)}}_{1}\, dt = R\int_0^{2\pi} dt$$

$$= Rt\Big|_0^{2\pi} = 2\pi R$$

Basic idea of why this works.



(picture with 4 subdivisions)

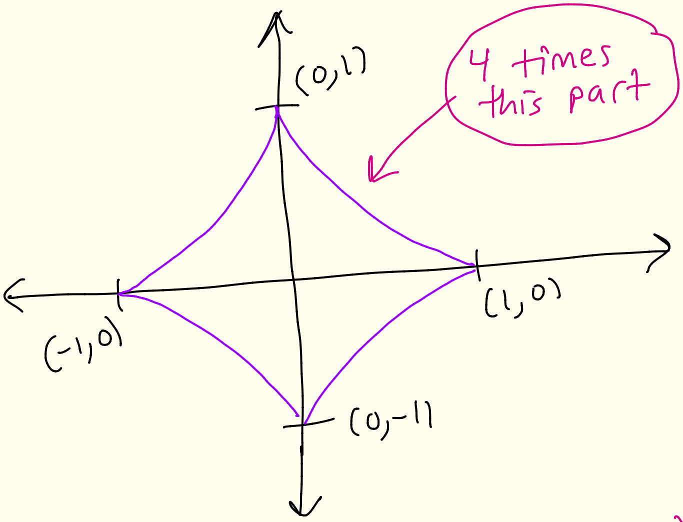
You can appoximate the length of the curve by breaking it into straight line segments.

If you add up the lengths of the straight lines then you get an approximation of the arclength of the curve.

If you then take the limit as the number of subdivisions goes to ∞ you'll get the arclength formula on page 1.

Ex: Find the length of the hypocycloid given by

$\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$, $0 \le t \le 2\pi$.



$$\vec{r}\,'(t) = \langle 3\cos^2(t)\cdot(-\sin(t)),\ 3\sin^2(t)\cdot\cos(t) \rangle$$

$$\text{arclength} = 4\int_0^{\pi/2} |\vec{r}\,'(t)|\,dt =$$

$$= 4\int_0^{\pi/2} \sqrt{\left(3\cos^2(t)\cdot(-\sin(t))\right)^2 + \left(3\sin^2(t)\cos(t)\right)^2}\,dt$$

$$= 4\int_0^{\pi/2} \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)}\,dt$$

$$= 4\int_0^{\pi/2} \sqrt{9\cos^2(t)\sin^2(t)\underbrace{[\cos^2(t)+\sin^2(t)]}_{1}}\,dt$$

$$= 4\int_0^{\pi/2} \sqrt{9\cos^2(t)\sin^2(t)}\,dt$$

$$= 4\int_0^{\pi/2} 3\,|\cos(t)|\,|\sin(t)|\,dt$$

$$\boxed{\sqrt{x^2} = |x|}$$

$$\boxed{\begin{array}{l|l} u = \sin(t) & t = 0 \to u = \sin(0) = 0 \\ du = \cos(t)\,dt & t = \pi/2 \to u = \sin(\pi/2) = 1 \end{array}}$$

$$= 12\int_0^{\pi/2} \underbrace{\cos(t)\,\underbrace{\sin(t)}_{u}\,dt}_{du} = 12\int_0^1 u\,du$$

$$= 12\,\frac{u^2}{2}\Big|_0^1 = 12\left[\frac{1^2}{2} - \frac{0^2}{2}\right]$$

$$= 6$$

when $0 \le t \le \pi/2$

$0 \le \sin(t)$

$0 \le \cos(t)$

So, $|\sin(t)| = \sin(t)$

$|\cos(t)| = \cos(t)$

when $0 \le t \le \pi/2$

arclength of entire curve (all 4 parts)