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MATH 540A
Due: 10/24/07

3.3 - A, B, C

$\boxed{3.3}$ [A and B: Find groups that the following are isomorphic to and use the first isomorphism theorem to prove it!]

A) [$\mathbb{Z}_2\times\mathbb{Z}_4/\langle(0,1)\rangle$]

Let's see what this group looks like. Let $H=\langle(0,1)\rangle=\{(0,1),(0,2),(0,3),(0,0)\}$. Then we pick $(1,0)\notin H$ to get the following left coset:
$(1,0)+H=\{(1,1),(1,2),(1,3),(1,0)\}$
Since $\left|\mathbb{Z}_2\times\mathbb{Z}_4/\langle(0,1)\rangle\right| = \frac{|\mathbb{Z}_2\times\mathbb{Z}_4|}{|\langle(0,1)\rangle|}=\frac{8}{4}=2$, we know that we have all of the left cosets. So,

$$\mathbb{Z}_2\times\mathbb{Z}_4/\langle(0,1)\rangle = \{H, (1,0)+H\}.$$

Since the order of this quotient group is 2, and 2 is prime, we know that it is isomorphic to $\mathbb{Z}_2$.

To use the first isomorphism theorem, we need a function (call it ϕ) from $\mathbb{Z}_2\times\mathbb{Z}_4$ to $\mathbb{Z}_2$ such that $\ker\phi=\langle(0,1)\rangle$, and that ϕ is a homomorphism and is onto. So, let $\phi((a,b))=a$. Then,

$$\ker\phi=\{(a,b)\in\mathbb{Z}_2\times\mathbb{Z}_4 \mid a=0\}=\{(0,b)\in\mathbb{Z}_2\times\mathbb{Z}_4\}=\langle(0,1)\rangle.$$

$\phi((a,b)+(c,d))=\phi((a+c,b+d))=a+c=\phi((a,b))+\phi((c,d))$, $\forall a,c\in\mathbb{Z}_2$, $b,d\in\mathbb{Z}_4$, so ϕ is a homomorphism.

Let $x\in\mathbb{Z}_2$. Then $\phi((x,0))=x$, so ϕ is onto. (so $\phi(\mathbb{Z}_2\times\mathbb{Z}_4)=\mathbb{Z}_2$)

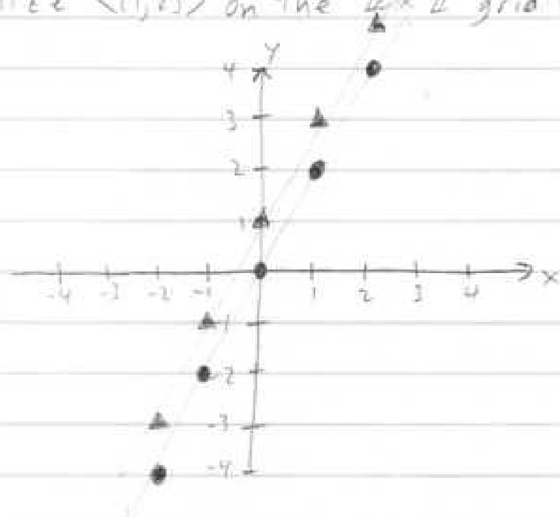
Thus, by the first isomorphism theorem,

$$\mathbb{Z}_2\times\mathbb{Z}_4/\langle(0,1)\rangle\cong\mathbb{Z}_2. \quad ▨$$

(3.3 cont) B) $\left[\mathbb{Z}\times\mathbb{Z}/\langle(1,2)\rangle\right]$

$\langle(1,2)\rangle = \{\ldots, (-2,-4), (-1,-2), (0,0), (1,2), (2,4), \ldots\}$

Let's visualize $\langle(1,2)\rangle$ on the $\mathbb{Z}\times\mathbb{Z}$ grid (by • dots):



It's points on the line $y=2x$. Note that if we shift the "line" up and down, we hit all of $\mathbb{Z}\times\mathbb{Z}$. Also, each "line" will be disjoint with other "lines". Infact, each "line" is a coset. For example, $(0,1)+\langle(1,2)\rangle = \{\ldots, (-2,-3), (-1,-1), (0,1), (1,3), (2,5), \ldots\}$ is drawn above using ▲ triangles. In general, a coset will be $(0,n)+\langle(1,2)\rangle$ for $n\in\mathbb{Z}$. So,

$$\mathbb{Z}\times\mathbb{Z}/\langle(1,2)\rangle = \{(0,n)+\langle(1,2)\rangle \mid n\in\mathbb{Z}\}.$$

This lends itself to a natural isomorphism with $\mathbb{Z}$, where each coset is paired with it's y-intercept.

see next page

DB3

(3.3 cont) (B cont)

To use the first isomorphism theorem, however, we need a function, call it ϕ, from $\mathbb{Z}\times\mathbb{Z}$ to $\mathbb{Z}$. We also need that $\ker\phi = \langle(1,2)\rangle$ and that ϕ is a homomorphism and is onto.

So, define $\phi((a,b)) = \overbrace{b-2a}^{\text{(note that this is the y-intercept of the line with slope 2 passing through (a,b))}}$. Well,

$$\ker\phi = \{(a,b)\in\mathbb{Z}\times\mathbb{Z} \mid b-2a=0\} = \{(a,2a)\in\mathbb{Z}\times\mathbb{Z}\} = \langle(1,2)\rangle.$$

$$\begin{aligned}\phi((a,b)+(c,d)) &= \phi((a+c,b+d)) \\ &= b+d-2(a+c) \\ &= b-2a+d-2c \\ &= \phi((a,b))+\phi((c,d)), \quad \forall a,b,c,d\in\mathbb{Z},\end{aligned}$$

so ϕ is a homomorphism.

Let $x\in\mathbb{Z}$. Then $\phi((-x,-x)) = -x+2x = x$. So, ϕ is onto. (so $\phi(\mathbb{Z}\times\mathbb{Z})=\mathbb{Z}$)

Thus, by the first isomorphism theorem,

$$\mathbb{Z}\times\mathbb{Z}/\langle(1,2)\rangle \cong \mathbb{Z}. \quad \blacksquare$$

(3.3 cont) c) [Let G and G' be groups, and let H and H' be normal subgroups of G and G', respectively. Let f be a homomorphism from G to G'. Show that f induces a natural homomorphism, $g: G/H \to G'/H'$ if $f(H)$ is a subset of H'.]

Pf: Let $g(xH) = f(x)H'$. Let's show that g is well-defined and is a homomorphism.

g is well-defined: Let $xH, yH \in G/H$ s.t. $xH = yH$. Here, $x, y \in G$. Let's show that $x^{-1}y \in H$. Since $xH = yH$, we get $H = x^{-1}xH = x^{-1}HxH = x^{-1}HyH = x^{-1}yH$. So, $x^{-1}y \in H$. So,

$$g(yH) = f(y)H' = f(xx^{-1}y)H' = f(x)f(x^{-1}y)H' = f(x)H'f(x^{-1}y)H' = f(x)H'H' = f(x)H' = g(xH).$$

(Since f is a homomorphism) (Since $x^{-1}y \in H$, so $f(x^{-1}y) \in f(H) \subset H'$)

g is a homomorphism: Let $xH, yH \in G/H$. Then,

$$\begin{aligned} g(xHyH) &= g(xyH) \\ &= f(xy)H' \\ &= f(x)f(y)H' \quad \text{since } f \text{ is a homomorphism} \\ &= f(x)H'f(y)H' \\ &= g(xH)g(yH) \end{aligned}$$

Thus, g is a well-defined homomorphism from G/H to G'/H'. ▨