Math 4570

8/24/20

1. Class recordings will be put on canvas.

2. Class lecture notes will be on the course website.

3. I'm using your calstatela email to send announcements to the class. If you want me to use a different email then let me know. You can just email me with 4570 and the email you want me to use.

Def: A **field** F is a set with two binary operations denoted by $+$ and $\cdot$, such that the following are true.

(F1) For all $a, b \in F$ there exist unique elements $a+b$ and $a \cdot b$ in F.

(F2) For all $a, b, c \in F$ we have

$a+b = b+a$ $a \cdot b = b \cdot a$ (commutative properties)	$a+(b+c) = (a+b)+c$ $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ (associative properties)	$a\cdot(b+c) = a\cdot b + a\cdot c$ $(b+c)\cdot a = b\cdot a + c\cdot a$ (distributive properties)

(F3) There exist elements 0 and 1 in F where $a+0 = 0+a = a$ and $a \cdot 1 = 1 \cdot a = a$ for all $a \in F$

(F4) For each $a \in F$, there exists $d \in F$ with $a+d = d+a = 0$.

(F5) For each $a \in F$, where $a \neq 0$, then there exists $f \in F$ with $a \cdot f = f \cdot a = 1$.

<u>HW:</u> 0, 1, d, f from F3 / F4 / F5 are unique.

We call 0 the <u>additive identity of F</u>.

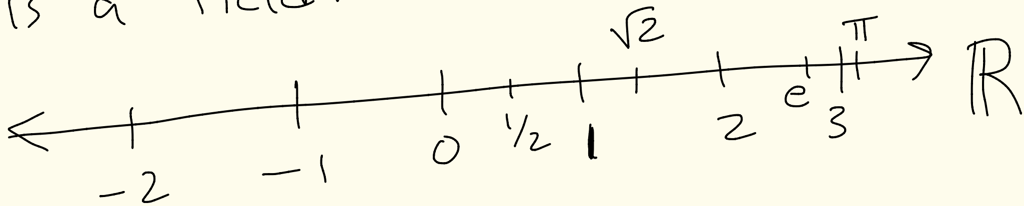
We call 1 the <u>multiplicative identity of F</u>.

We denote d in F4 as $-a$ and call it the <u>additive inverse of a</u>.

We denote f in F5 as a^{-1} and call it the <u>multiplicative inverse of a</u>.

$\boxed{\text{Ex:}}$

The set of real numbers $\mathbb{R}$ is a field.



$a = 5, \; -a = -5$ (additive inverse)

$a = 2, \; a^{-1} = \frac{1}{2}$ (multiplicative inverse)

$\boxed{\text{Ex:}}$ The set of rational numbers $\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z}, \; b \neq 0 \right\}$

$= \left\{ 1, 0, 5, \frac{1}{2}, \frac{10}{3}, -\frac{7}{10}, \ldots \right\}$

is a field.

$a = \frac{10}{3}, \; \underbrace{-a = \frac{-10}{3}}_{\text{additive inverse}}, \; \underbrace{a^{-1} = \frac{3}{10}}_{\text{multiplicative inverse}}$

Ex: The complex numbers

$$\mathbb{C} = \{ x + iy \mid x, y \in \mathbb{R} \}$$

(where $i^2 = -1$) is a field.

Ex: $\mathbb{Z}_p = \{ \bar{0}, \bar{1}, \bar{2}, \ldots, \overline{p-1} \}$ is a field where p is prime

$\mathbb{Z}_p$ is set of integers modulo p.

Def: Let F be a field.

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A vector space over F is a set V with two operations. The first operation is addition which takes two elements $v_1, v_2 \in V$ and produces a unique element $v_1 + v_2 \in V$.

V is sometimes called the set of "vectors"

The second operation is scalar multiplication, which takes one element $a \in F$ and one element $v \in V$ and produces a unique element $av \in V$.

Can also write $a \cdot v$

The following properties must hold.

(V1) For all $v_1, v_2 \in V$ we have $v_1 + v_2 = v_2 + v_1$.

(V2) For all $v_1, v_2, v_3 \in V$ we have $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$.

(V3) There exists an element $\vec{0}$ in V where $\vec{0} + v = v + \vec{0} = v$ for all $v \in V$.

(V4) For each $v \in V$ there exists $w \in V$ with $v+w=w+v=\vec{0}$.

(V5) For each $v \in V$, we have $1v = v$. [Here 1 is from F]

(V6) For $a, b \in F$ and $v \in V$ we have $(ab)v = a(bv)$.

(V7) For all $a \in F$ and $v_1, v_2 \in V$ we have $a(v_1 + v_2) = av_1 + av_2$.

(V8) For all $a, b \in F$ and $v \in V$ we have $(a+b)v = av + bv$

Later we will show that $\vec{0}$ from (V3) and w from (V4) are unique.

$\vec{0}$ is called the <u>zero vector</u> of V.

w is called the <u>additive inverse</u> of v and is denoted by $-v$.

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Ex: $F = \mathbb{R}$, $V = \mathbb{R}^2$

then $V = \mathbb{R}^2$ is a vector space over $F = \mathbb{R}$ using the operations

$$(x, y) + (a, b) = (x + a, y + b)$$ vector addition

$$\alpha (x, y) = (\alpha x, \alpha y)$$ scalar mult.

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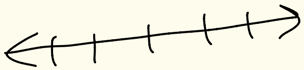
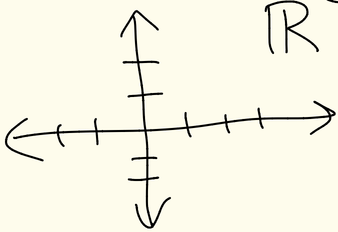
field



$\mathbb{R}$

vectors



$\mathbb{R}^2$

$$(1, 2) + (-1, 5) = (0, 7)$$

$$-10(1, 3) = (-10, -30)$$

Ex: Let F be a field.
Let $V = F^n$ where n is an integer $n \geq 1$.
Then $V = F^n$ is a vector space over F using the following operations.

Let $x = (a_1, a_2, \ldots, a_n)$
and $y = (b_1, b_2, \ldots, b_n)$ be in $V = F^n$.

and $\alpha \in F$.
Vector addition will be defined as
$$x + y = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
and scalar multi. will be defined as
$$\alpha x = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

Pg 3

proof:

Let $x, y, z \in V = F^n$ where $x = (a_1, a_2, \dots, a_n)$, $y = (b_1, b_2, \dots, b_n)$ and $z = (c_1, c_2, \dots, c_n)$.

Let $a, b \in F$.

V1 We have

$$\begin{aligned} x + y &= (a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) \\ &= (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \\ &= (b_1 + a_1, b_2 + a_2, \dots, b_n + a_n) \\ &= (b_1, b_2, \dots, b_n) + (a_1, a_2, \dots, a_n) \\ &= y + x. \end{aligned}$$

def of +

F is a field $a + b = b + a$ $\forall a, b \in F$ (F2)

V2 We have

$$
\begin{aligned}
&(x+y)+z \\
&= \left[(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)\right] + (c_1, c_2, \ldots, c_n) \\
&\overset{\text{def of } +}{=} (a_1+b_1, a_2+b_2, \ldots, a_n+b_n) + (c_1, c_2, \ldots, c_n) \\
&\overset{\text{def of } +}{=} ((a_1+b_1)+c_1, (a_2+b_2)+c_2, \ldots, (a_n+b_n)+c_n) \\
&\overset{F \text{ is associative } F2}{=} (a_1+(b_1+c_1), a_2+(b_2+c_2), \ldots, a_n+(b_n+c_n)) \\
&\overset{\text{def of } +}{=} (a_1, a_2, \ldots, a_n) + (b_1+c_1, b_2+c_2, \ldots, b_n+c_n) \\
&\overset{\text{def of } +}{=} (a_1, a_2, \ldots, a_n) + \left[(b_1, b_2, \ldots, b_n) + (c_1, c_2, \ldots, c_n)\right] \\
&= x+(y+z)
\end{aligned}
$$

pg 5

V3 Let $\vec{0} = (0, 0, \ldots, 0)$
where 0 is the zero element of F.

Then

$$\vec{0} + x = (0, 0, \ldots, 0) + (a_1, a_2, \ldots, a_n)$$
$$= (0 + a_1, 0 + a_2, \ldots, 0 + a_n)$$
$$= (a_1, a_2, \ldots, a_n)$$
$$= x$$

and

$$x + \vec{0} = (a_1, a_2, \ldots, a_n) + (0, 0, \ldots, 0)$$
$$= (a_1 + 0, a_2 + 0, \ldots, a_n + 0)$$
$$= (a_1, a_2, \ldots, a_n)$$
$$= x$$

Thus, $x + \vec{0} = \vec{0} + x = x$.

V4 Given $x=(a_1, a_2, \ldots, a_n)$
define w to be $w=(-a_1, -a_2, \ldots, -a_n)$
[Note that w exists because given $a \in F$ there exists $-a \in F$.]

Then,

$$x+w = (a_1, a_2, \ldots, a_n) + (-a_1, -a_2, \ldots, -a_n)$$
$$= (a_1 - a_1, a_2 - a_2, \ldots, a_n - a_n)$$
$$= (0, 0, \ldots, 0) = \vec{0}$$

Similarly,

$$w+x = (-a_1, -a_2, \ldots, -a_n) + (a_1, a_2, \ldots, a_n)$$
$$= (-a_1 + a_1, -a_2 + a_2, \ldots, -a_n + a_n)$$
$$= (0, 0, \ldots, 0) = \vec{0}.$$

So given $x \in V$, there exists $w \in V$ where $x + w = w + x = \vec{0}$.

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V5 We have

$$1x = 1(a_1, a_2, \ldots, a_n)$$
$$= (1a_1, 1a_2, \ldots, 1a_n)$$
$$= (a_1, a_2, \ldots, a_n) = x$$

def of scalar mult

[Here 1 is the multiplicative identity of F]

V6 We have

$$(ab)x = (ab)(a_1, a_2, \ldots, a_n)$$
$$= ((ab)a_1, (ab)a_2, \ldots, (ab)a_n)$$
$$= (a(ba_1), a(ba_2), \ldots, a(ba_n))$$
$$= a((ba_1), (ba_2), \ldots, (ba_n))$$
$$= a[b(a_1, a_2, \ldots, a_n)]$$
$$= a[bx]$$

def of scalar mult.

F is associative

def of scalar mult

V7 We have

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$$a(x+y) = a\left[(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n)\right]$$

def of +

$$= a(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$$

def of scalar mult

$$= (a(a_1+b_1), a(a_2+b_2), \ldots, a(a_n+b_n))$$

F has the distributive property

$$= (aa_1+ab_1, aa_2+ab_2, \ldots, aa_n+ab_n)$$

def of +

$$= (aa_1, aa_2, \ldots, aa_n) + (ab_1, ab_2, \ldots, ab_n)$$

def of scalar mult

$$= a(a_1, a_2, \ldots, a_n) + a(b_1, b_2, \ldots, b_n)$$

$$= ax + ay$$

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V8 We have

$$(a+b)x = (a+b)(a_1, a_2, \ldots, a_n)$$

def of scalar mult

$$= ((a+b)a_1, (a+b)a_2, \ldots, (a+b)a_n)$$

F has distributive property

$$= (aa_1 + ba_1, aa_2 + ba_2, \ldots, aa_n + ba_n)$$

def of +

$$= (aa_1, aa_2, \ldots, aa_n) + (ba_1, ba_2, \ldots, ba_n)$$

def of scalar mult

$$= a(a_1, a_2, \ldots, a_n) + b(a_1, a_2, \ldots, a_n)$$

$$= ax + bx$$

Since V1 – V8 are true $V = F^n$ is a vector space over F

$\blacksquare$

Ex: $\mathbb{R}^3$ is a vector space over $\mathbb{R}$ (pg 10)

$\mathbb{C}^{10}$ is a vector space over $\mathbb{C}$

Ex: Let F be a field.
Let $V = M_{m,n}(F)$ be the set of all $m \times n$ matrices with entries from F. Then V is a vector space over F where vector addition is defined as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11}, & a_{12}+b_{12}, & \cdots, & a_{1n}+b_{1n} \\ a_{21}+b_{21}, & a_{22}+b_{22}, & \cdots, & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1}, & a_{m2}+b_{m2}, & \cdots, & a_{mn}+b_{mn} \end{pmatrix}$$

and scalar multiplication is defined as

$$\alpha \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

Here $\vec{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$.

<u>Ex:</u> $F = \mathbb{R}$

$V = M_{2,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$

$\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

<u>Ex:</u> $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 10 \end{pmatrix}$

$10 \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 10 \\ 30 & 50 \end{pmatrix}$

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Ex: Let $F = \mathbb{R}$ or $F = \mathbb{C}$.
Let $n \geq 0$ be an integer.
Define the set $P_n(F)$ to be the set of all polynomials with coefficents from F of degree less than or equal to n.
So,

$$P_n(F) = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \,\middle|\, a_0, a_1, a_2, \ldots, a_n \in F \right\}$$

Then $V = P_n(F)$ is a vector space over F where vector addition is given by

$$(a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n)$$
$$= (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n$$

and scalar multiplication is defined as

$$\alpha\left[a_0 + a_1 x + \cdots + a_n x^n\right] = (\alpha a_0) + (\alpha a_1) x + \cdots + (\alpha a_n) x^n$$

pg 2

Here the zero vector is

$$\vec{0} = 0 + 0x + 0x^2 + \cdots + 0x^n$$

We define equality as follows:

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

and $g(x) = b_0 + b_1 x + \cdots + b_n x^n$

We define $f = g$ iff

$a_0 = b_0$, $a_1 = b_1$, $\ldots$, $a_n = b_n$

Ex:

$$P_3(\mathbb{R}) = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

zero vector

$$= \{ 0, 1, 1+x, 1+x^3, \pi + ex^2 + x^3, 10x^3 + 1, \ldots \}$$

$$(1+x^3) + (10x^3 + 1) = 2 + 11x^3$$

$$10(1+x^3) = 10 + 10x^3$$

<u>Theorem:</u> Let V be a vector space over a field F.

① The element $\vec{0}$ from V3 is unique. [where $\vec{0} + x = x + \vec{0} = x$ for all $x \in V$]

② Given $x \in V$, the element $w \in V$ from V4 where $x + w = w + x = \vec{0}$ is unique. [From now on we will write $-x$ for w].

<u>proof:</u>

① Suppose $\vec{0}_1, \vec{0}_2 \in V$ where $\vec{0}_1 + x = x + \vec{0}_1 = x$ and $\vec{0}_2 + x = x + \vec{0}_2 = x$ for all $x \in V$. Then

$$\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2$$

$x = x + \vec{0}_2$
$x = \vec{0}_1$

$\vec{0}_1 + x = x$
$x = \vec{0}_2$

So, $\vec{0}_1 = \vec{0}_2$.

pg 4

(2) Let $x \in V$.

Suppose $w_1, w_2 \in V$
where $w_1 + x = x + w_1 = \vec{0}$
and $w_2 + x = x + w_2 = \vec{0}$.

We have $x + w_1 = \vec{0}$.
Adding w_2 to both sides gives

$$\underbrace{w_2 + (x + w_1)}_{\text{regroup with associativity}} = \underbrace{w_2 + \vec{0}}_{w_2}.$$

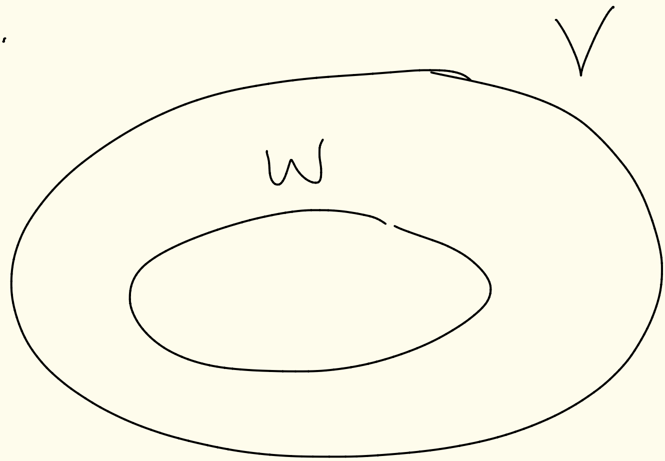
So,

$$\underbrace{(w_2 + x)}_{\vec{0}} + w_1 = w_2.$$

Thus, $\vec{0} + w_1 = w_2$.

It follows that $w_1 = w_2$. ∎

Def: Let V be a vector space over a field F. Let W be a subset of V. We say that W is a subspace of V if W is a vector space over F using the same vector addition and scalar multiplication as in V.



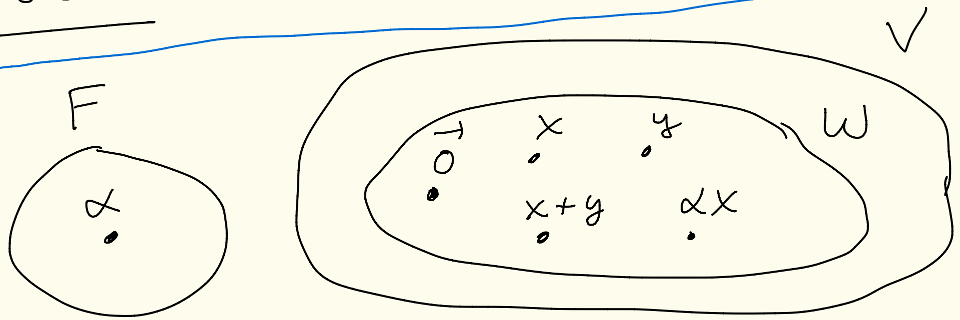
Theorem: Let V be a vector space over a field F. Let W be a subset of V.
W is a subspace of V iff the following conditions hold:

① $\vec{0} \in W$

② If $x, y \in W$, then $x + y \in W$. — W is closed under addition

③ If $\alpha \in F$ and $x \in W$, then $\alpha x \in W$. — W is closed under scalar mult.

proof: HW.



Ex: $V = \mathbb{R}^3$ over $F = \mathbb{R}$.

Let $W = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$

Is W a subspace of V?

$$\boxed{W = \{(1,5,0), (\pi,1,0), (3,2,0), (1,2,0), \ldots\}}$$

① Set $a = 0, b = 0$ then we get $\vec{0} = (0,0,0) \in W$.

② Let $x = (a_1, b_1, 0)$ and $y = (a_2, b_2, 0)$ be in W.

Then $x + y = (a_1 + a_2, b_1 + b_2, 0) \in W$

③ Let $\alpha \in \mathbb{R}$ and $x = (a_1, b_1, 0) \in W$.

Then, $\alpha x = (\alpha a_1, \alpha b_1, 0) \in W$.

By ①/②/③, W is a subspace of V.

Ex: Let

$$V = M_{2,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$$

and $F = \mathbb{R}$.

Let $W = \left\{ \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

Is W a subspace of V?

$$\boxed{W = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ldots \right\}}$$

not 1

① $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin W$. So, W is not a subspace.

not 1

② $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in W$ but

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \notin W$$

So, W is not closed under $+$,

③ W isn't closed under scalar mult, since $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in W$ but $5\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 0 & 5 \end{pmatrix} \notin W$.

So, W is not a subspace of V. You can use any of ① – ③ not holding to show this.

Note: Let V be a vector space over a field F. V has at least these subspaces:

$$W = \{\vec{0}\}$$ ← the trivial subspace

$$W = V$$

Done with HW 1 stuff.
Now starting HW 2 stuff.

Bases of vector spaces

<u>Def:</u> Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n \in V$.

① The <u>span</u> of the vectors $v_1, v_2, \ldots, v_n$ is defined to be the set

$$\text{span}(\{v_1, v_2, \ldots, v_n\}) = \{c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \mid c_1, c_2, \ldots, c_n \in F\}$$

② If $V = \text{span}(\{v_1, v_2, \ldots, v_n\})$ we say that $v_1, v_2, \ldots, v_n$ span V, or we say $v_1, v_2, \ldots, v_n$ form a <u>spanning set</u> for V.

③ The expression $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$ is called a <u>linear combination</u> of $v_1, v_2, \ldots, v_n$.

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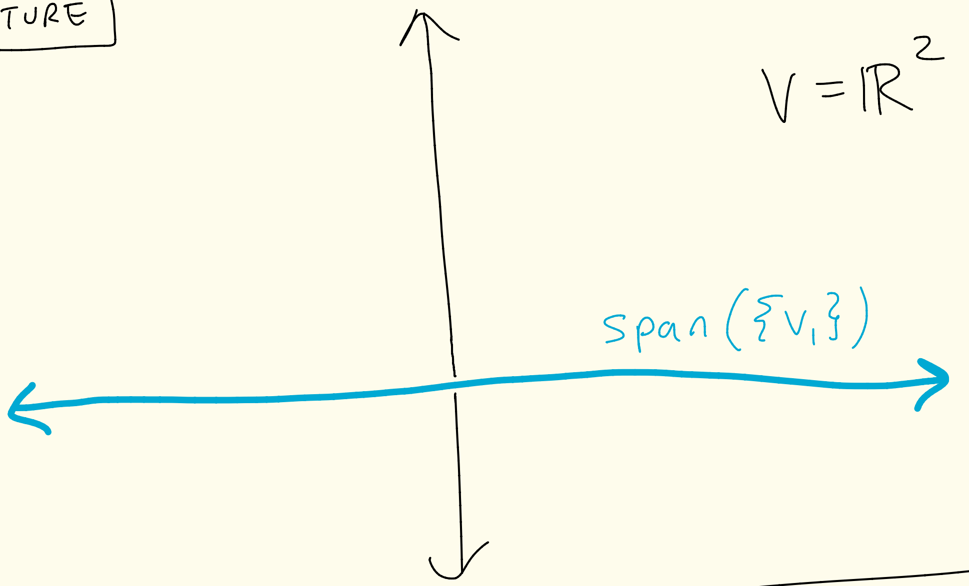
Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $v_1 = (1, 0)$.

$$\begin{aligned} \mathrm{span}(\{v_1\}) &= \{c_1 v_1 \mid c_1 \in \mathbb{R}\} \\ &= \{c_1 (1,0) \mid c_1 \in \mathbb{R}\} \\ &= \{(c_1, 0) \mid c_1 \in \mathbb{R}\} \end{aligned}$$

PICTURE



v_1 does not span $\mathbb{R}^2$.

v_1 spans the x-axis.

<u>Ex:</u> $V = \mathbb{R}^2$, $F = \mathbb{R}$

$v_1 = (1,0)$, $v_2 = (0,1)$

$$\begin{aligned}
\operatorname{span}(\{v_1, v_2\}) &= \{c_1(1,0) + c_2(0,1) \mid c_1, c_2 \in \mathbb{R}\} \\
&= \{(c_1, 0) + (0, c_2) \mid c_1, c_2 \in \mathbb{R}\} \\
&= \{(c_1, c_2) \mid c_1, c_2 \in \mathbb{R}\} \\
&= \mathbb{R}^2
\end{aligned}$$

So, v_1, v_2 span $\mathbb{R}^2$.

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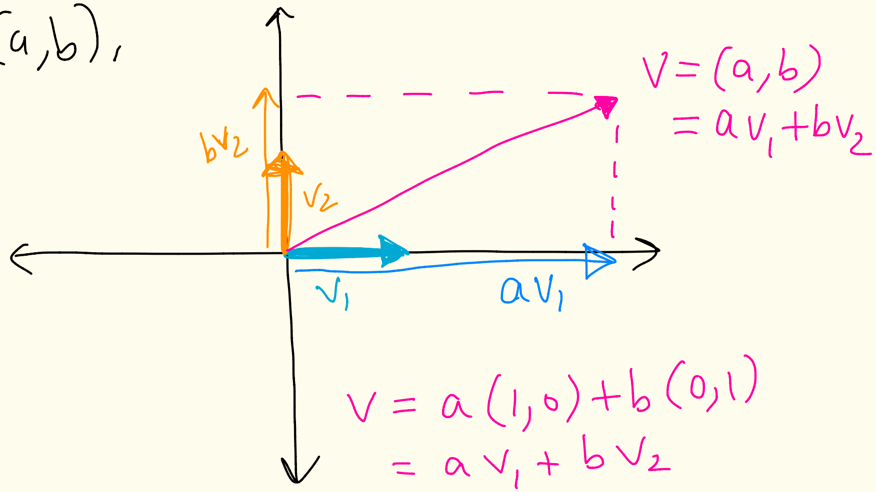
Last time we showed that $v_1 = (1,0)$ and $v_2 = (0,1)$ span $\mathbb{R}^2$.

Let $v = (a,b)$,



$$v = a(1,0) + b(0,1)$$
$$= av_1 + bv_2$$

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $v = (2,1)$ and $w = (-1,1)$.

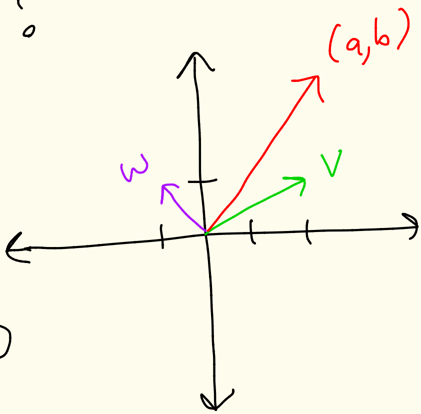
Do v, w span $\mathbb{R}^2$?

Let $(a,b) \in \mathbb{R}^2$.

The question is:

Can we always find $c_1, c_2 \in \mathbb{R}^2$ where

$$(a,b) = c_1 v + c_2 w \quad ?$$



That is we want to see if we can always solve

$$\begin{pmatrix} a \\ b \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{v} + c_2 \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{w}$$

We have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2c_1 - c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} a &= 2c_1 - c_2 \\ b &= c_1 + c_2 \end{aligned}}$$

3 operations in Gaussian Elimination

1. interchange two rows
2. multiply a row by a non-zero constant
3. Add a constant multiple of one row to another row

$$\left(\begin{array}{cc|c} 2 & -1 & a \\ 1 & 1 & b \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 2 & -1 & a \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & -3 & a-2b \end{array}\right) \xrightarrow{-\frac{1}{3}R_2 \to R_2}$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & 1 & \frac{a-2b}{-3} \end{array}\right)$$

$$\boxed{\begin{aligned} c_1 + c_2 &= b \\ c_2 &= \frac{a-2b}{-3} = -\frac{1}{3}a + \frac{2}{3}b \end{aligned}}$$

$$c_1 = b - c_2 = b - \left(-\frac{1}{3}a + \frac{2}{3}b\right) = \frac{1}{3}a + \frac{1}{3}b$$

$$c_2 = -\frac{1}{3}a + \frac{2}{3}b$$

So we can solve the system no matter what (a,b) is. We have

$$\begin{aligned} (a,b) &= c_1 v + c_2 w \\ &= \left(\frac{1}{3}a + \frac{1}{3}b\right)v + \left(-\frac{1}{3}a + \frac{2}{3}b\right)w \end{aligned}$$

Thus, $\operatorname{span}(\{v, w\}) = \mathbb{R}^2$

(HW 1 #4a)

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Lemma: Let V be a vector space over a field F. Let $\vec{0}$ be the zero vector of V and 0 be the zero element of F. Then $0w = \vec{0}$ for all $w \in V$.

proof: We have that

$$0w = (0+0)w = 0w + 0w$$

Add $-(0w)$ to both sides to get

$$\underbrace{-0w + 0w}_{\vec{0}} = \underbrace{-0w + 0w}_{\vec{0}} + 0w.$$

So, $\vec{0} = \vec{0} + 0w$.

Thus, $\vec{0} = 0w$. $\blacksquare$

Theorem: Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n \in V$.
Let $W = \text{span}(\{v_1, v_2, \ldots, v_n\})$

$$= \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \mid c_1, c_2, \ldots, c_n \in F\}$$

Then:

① W is a subspace of V.

② If U is any subspace of V that contains $v_1, v_2, \ldots, v_n$ [that is, if $v_1, v_2, \ldots, v_n \in U$], then $W \subseteq U$.

---

proof:

① Let's show W is a subspace of V.

(i) $\vec{0} = \underbrace{\vec{0} + \vec{0} + \cdots + \vec{0}}_{n \text{ times}}$

$$= 0v_1 + 0v_2 + \cdots + 0v_n \in W$$

(ii) Let $x, y \in W$.

Then $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

and $y = d_1 v_1 + d_2 v_2 + \cdots + d_n v_n$

where $c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n \in F$

Then,

$$x + y = (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \cdots$$
$$\cdots + (c_n + d_n) v_n \in W$$

(iii) Let $\alpha \in F$ and $z \in W$.

Then, $z = e_1 v_1 + e_2 v_2 + \cdots + e_n v_n$

where $e_1, e_2, \ldots, e_n \in F$.

So,

$$\alpha z = \alpha \left[e_1 v_1 + e_2 v_2 + \cdots + e_n v_n \right]$$
$$= \alpha(e_1 v_1) + \alpha(e_2 v_2) + \cdots + \alpha(e_n v_n)$$
$$= (\alpha e_1) v_1 + (\alpha e_2) v_2 + \cdots + (\alpha e_n) v_n \in W$$

So, by (i), (ii), (iii)
W is a subspace of V.

② Suppose U is a subspace of V
and $v_1, v_2, \ldots, v_n \in U$.
Let's show $W \subseteq U$.
Let $x \in W$.
Then $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$
where $c_1, c_2, \ldots, c_n \in F$.

Since $v_1, v_2, \ldots, v_n \in U$ and U is a subspace we have that
$c_1 v_1, c_2 v_2, \ldots, c_n v_n \in U$.

U is closed under scalar mult.

Since $c_1 v_1, c_2 v_2, \ldots, c_n v_n \in U$ and U is a subspace,
$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \in U$

U is closed under $+$

Thus, $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \in U$.
So, $W \subseteq U$. ∎

Def: Let V be a vector space over a field F.

Let $v_1, v_2, \ldots, v_n \in V$.

We say that $v_1, v_2, \ldots, v_n$ are <u>linearly dependent</u> if there exist $c_1, c_2, \ldots, c_n \in F$ (not all equal to zero)

such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$$

If $v_1, v_2, \ldots, v_n$ are not linearly dependent then we call them <u>linearly independent</u>.

Note: You can always write

$$0 v_1 + 0 v_2 + \cdots + 0 v_n = \vec{0}$$

To be lin. dep. means you can write $\vec{0}$ in more than one way in the form $\vec{0} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

Ex: Are these vectors lin. ind. or lin. dep. in $\mathbb{R}^3$?

$v_1 = (1,0,1)$, $v_2 = (-1,2,1)$, $v_3 = (0,2,2)$

Let's see if we can solve

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

We have

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - c_2 \\ 2c_2 + 2c_3 \\ c_1 + c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Finish next time.

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(Continued from last week)

Ex: Are these vectors lin. ind. or lin. dep. in $\mathbb{R}^3$?

$v_1 = (1,0,1)$, $v_2 = (-1,2,1)$, $v_3 = (0,2,2)$

Let's see if we can solve

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

We have

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - c_2 \\ 2c_2 + 2c_3 \\ c_1 + c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{-R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ 2c_2 + 2c_3 &= 0 \end{aligned}}$$

$\leftarrow c_1 = c_2 = -c_3$

$\leftarrow c_2 = -c_3$

Solutions:

$$\boxed{\begin{aligned} c_1 &= -t \\ c_2 &= -t \qquad t \in \mathbb{R} \\ c_3 &= t \end{aligned}}$$

So, $c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$

has solutions

$$-t v_1 - t v_2 + t v_3 = \vec{0}$$

for any $t \in \mathbb{R}$.

So, $t=1$ gives $-v_1 - v_2 + v_3 = \vec{0}$

The vectors are linearly dependent.

You have

$$v_3 = v_1 + v_2$$

← ("dependency" equation)

ie you can write one of the vectors as a linear combo of the other vectors

---

Ex: Let

$$V = P_2(\mathbb{C}) = \{a + bx + cx^2 \mid a,b,c \in \mathbb{C}\}$$

$$F = \mathbb{C}$$

Are the following vectors linearly independent or dependent?

$v_1 = 1 + 2x$

$v_2 = -i$

$v_3 = x$

$v_4 = 5 + ix^2$

Find the solutions to

$$c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 = \vec{0}$$

That is

$$c_1(1+2x) + c_2(-i) + c_3(x) + c_4(5+ix^2) = 0 + 0x + 0x^2$$

which becomes

$$c_1 + 2c_1 x - ic_2 + c_3 x + 5c_4 + ic_4 x^2 = 0 + 0x + 0x^2$$

which is

$$(c_1 - ic_2 + 5c_4) + (2c_1 + c_3)x + ic_4 x^2 = 0 + 0x + 0x^2$$

So,

$$\boxed{\begin{array}{l} c_1 - ic_2 + + 5c_4 = 0 \\ 2c_1 + c_3 = 0 \\ ic_4 = 0 \end{array}}$$

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$$\left(\begin{array}{cccc|c} 1 & -i & 0 & 5 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -i & 0 & 5 & 0 \\ 0 & 2i & 1 & -10 & 0 \\ 0 & 0 & 0 & i & 0 \end{array}\right)$$

$$\boxed{\begin{array}{l} i^2 = -1 \\ (-i)(i) = -i^2 = 1 \end{array}}$$

$$\xrightarrow[-i\,R_3 \to R_3]{-i * R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & -i & 0 & 5 & 0 \\ 0 & 2 & -i & 10i & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\boxed{\begin{array}{rcl} c_1 - i c_2 \qquad + 5c_4 &=& 0 \\ 2c_2 - ic_3 + 10ic_4 &=& 0 \\ c_4 &=& 0 \end{array}}$$

$$\begin{aligned} c_1 - i c_2 + 5 c_4 &= 0 \\ 2 c_2 - i c_3 + 10 i c_4 &= 0 \\ c_4 &= 0 \end{aligned}$$

$$c_4 = 0$$

$$\begin{aligned} c_1 - i c_2 &= 0 \\ 2 c_2 - i c_3 &= 0 \end{aligned}$$

$$\begin{aligned} c_3 &= t \\ c_2 &= \frac{i}{2} c_3 \\ &= \frac{i}{2} t \end{aligned}$$

$$\begin{aligned} c_1 &= i c_2 \\ &= i\left(\frac{i}{2} t\right) \\ &= \frac{i^2}{2} t \\ &= -\frac{1}{2} t \end{aligned}$$

Solutions:

$$c_1 = -\frac{1}{2} t$$

$$c_2 = \frac{i}{2} t \qquad t \in \mathbb{C}$$

$$c_3 = t$$

$$c_4 = 0$$

So,

$$-\frac{1}{2} t v_1 + \frac{i}{2} t v_2 + t v_3 + 0 v_4 = \vec{0}$$

$t = 2$ gives: $\boxed{-v_1 + i v_2 + 2 v_3 + 0 v_4 = \vec{0}}$

So, v_1, v_2, v_3, v_4 are lin. dep. in $P_2(\mathbb{C})$.

<u>Ex:</u> $V = \mathbb{R}^2$, $F = \mathbb{R}$

Are these vectors lin. dep. or lin. ind.?

$v_1 = (1, 0)$, $v_2 = (0, 1)$

Consider $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Then, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So, $c_1 = 0$ & $c_2 = 0$.

Thus, v_1 and v_2 are lin. independent.

<u>Def</u>: Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n \in V$. We say that $\beta = \{v_1, v_2, \ldots, v_n\}$ is a <u>basis</u> for V if

(1) $\text{span}(\beta) = V$

and

(2) β is a linearly independent set of vectors

<u>Ex</u>: $V = \mathbb{R}^2$ and $F = \mathbb{R}$
$v_1 = (1,0)$ and $v_2 = (0,1)$, $\beta = \{v_1, v_2\}$

$$\begin{aligned}\text{span}(\beta) &= \{c_1 v_1 + c_2 v_2 \mid c_1, c_2 \in \mathbb{R}\} \\ &= \left\{ c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\} = \mathbb{R}^2\end{aligned}$$

So, β spans $\mathbb{R}^2$.

Ex: $\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2v_1 - v_2$

We already checked that β is a linearly independent set.

So, $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $V = \mathbb{R}^2$ over $F = \mathbb{R}$

Ex: $V = M_{2,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}$

$F = \mathbb{R}$

$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Let $\beta = \{v_1, v_2, v_3, v_4\}$

Does β span $M_{2,2}(\mathbb{R})$?

Yes. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathbb{R})$.

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= a v_1 + b v_2 + c v_3 + d v_4$$

Is β a lin. ind. set?

$\vec{0}$

Suppose

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, $c_1 = c_2 = c_3 = c_4 = 0$ is the only solution.

So, β is a lin. ind. set.

$\boxed{\text{Thus, } \beta \text{ is a basis for } M_{2,2}(\mathbb{R}) \text{ over } \mathbb{R}.}$

Theorem Let V be a vector space over a field F. Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a subset of V. Then β is a basis for V iff every vector $x \in V$ can be expressed uniquely in the form

$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

where $c_1, c_2, \dots, c_n \in F$.

---

proof:

($\Rightarrow$) Suppose β is a basis for V.
Let $x \in V$.
Since β is a basis, β spans V.
So, there exist $c_1, c_2, \dots, c_n \in F$
where $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ (*)
We want to now show that
$c_1, c_2, \dots, c_n$ are unique.

Suppose we also have
$c_1', c_2', \ldots, c_n' \in F$ with

$$x = c_1' v_1 + c_2' v_2 + \cdots + c_n' v_n \quad (**)$$

Then $(*) - (**)$ gives

$$\vec{0} = (c_1 - c_1') v_1 + (c_2 - c_2') v_2 + \cdots + (c_n - c_n') v_n$$

Since β is a lin. ind. set
of vectors,

$$c_1 - c_1' = 0$$
$$c_2 - c_2' = 0$$
$$\vdots$$
$$c_n - c_n' = 0.$$

So, $c_1 = c_1'$, $c_2 = c_2'$, $\ldots$, $c_n = c_n'$.

So, the expression is unique.

$(\Longleftarrow)$ next time $\blacksquare$

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Theorem: Let V be a vector space over a field F. Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a subset of V. Then β is a basis for V iff every vector $x \in V$ can be written uniquely in the form $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ where $c_1, c_2, \dots, c_n \in F$.

proof: ($\Rightarrow$) last time.

($\Leftarrow$) Suppose every vector $x \in V$ can be written uniquely as $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ where $c_i \in V$.

This tells us that every $x \in V$ is in the span of β, so β spans V.

Why is β are lin. ind. set? pg 2

Suppose we want to solve

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$$

We definitely have this solution:

$$0 v_1 + 0 v_2 + \cdots + 0 v_n = \vec{0}.$$

But by assumption, there a unique solution. So the only solution to $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$ is $c_1 = c_2 = \cdots = c_n = 0$.

Thus, $\beta = \{v_1, v_2, \ldots, v_n\}$ is a linearly independent set.

So, β is a basis for V. ▨

Notation for the next Thm

Consider the system

$$\left.\begin{aligned} 10x_1 - 3x_2 + \tfrac{1}{3}x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}\right\} (*)$$

Let $A_1 = (10, -3, \frac{1}{3})$

$A_2 = (0, 5, -1)$

$A_3 = (-1, 1, 0)$

$X = (x_1, x_2, x_3)$

Then equations $(*)$ can be written

$$\left.\begin{aligned} A_1 \cdot X &= \vec{0} \\ A_2 \cdot X &= \vec{0} \\ A_3 \cdot X &= \vec{0} \end{aligned}\right\} \text{same as } (*)$$

Adding $\frac{1}{10}$ * row 1 to row 3 gives

$$10x_1 - 3x_2 + \frac{1}{3}x_3 = 0$$
$$5x_2 - x_3 = 0$$
$$\frac{7}{10}x_2 + \frac{1}{30}x_3 = 0$$

which can be represented by

$$A_1 \cdot X = \vec{0}$$
$$A_2 \cdot X = \vec{0}$$
$$\left(\frac{1}{10}A_1 + A_3\right) \cdot X = \vec{0}$$

Theorem: Let

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} (*)$$

be a system of m equations and n unknowns where $a_{ij} \in F$ for some field F. If $n > m$, then $(*)$ has a non-trivial solution. [That is, there is a solution $(x_1, x_2, \ldots, x_n) \in F^n$ with $(x_1, x_2, \ldots, x_n) \neq \vec{0}$.]

proof: [We follow the proof from Lang, Intro. to Linear Algebra, 2nd edition, pg 68-69]

We induct on m [the # of eqns].

Suppose $m = 1$.
So, $n > m = 1$. [ie $n \geq 2$ (at least 2 variables)]
So our system $(*)$ becomes

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

If $a_{11} = a_{12} = \dots = a_{1n} = 0$ then we get a non-trivial solution by setting $x_1 = x_2 = \dots = x_n = 1$.

Suppose one of the coefficients isn't 0.
Without loss of generality, assume $a_{11} \neq 0$.

Then the eqn $(*)$ is equivalent to

$$x_1 = -a_{11}^{-1}(a_{12}x_2 + \dots + a_{1n}x_n)$$

Set $x_2 = x_3 = \dots = x_n = 1$ and

$$x_1 = -a_{11}^{-1}(a_{12} + \dots + a_{1n}).$$

This gives a non-trivial solution.
So the base case $m=1$ is true.

Induction hypothesis

Now assume that the theorem is true for any system of $m-1$ equations with more than $m-1$ unknowns.

Suppose we have a system $(*)$ with m equations and n unknowns with $n > m > 1$.

If all the $a_{ij} = 0$ then
set $x_1 = x_2 = \dots = x_n = 1$
is a non-trivial solution.

Now suppose some coefficient $a_{\bar{i}\bar{j}} \neq 0$.
By renumbering the equations and variables we may assume $a_{11} \neq 0$.

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Set

$$A_1 = (a_{11}, a_{12}, \ldots, a_{1n})$$
$$A_2 = (a_{21}, a_{22}, \ldots, a_{2n})$$
$$\vdots \qquad\qquad \vdots$$
$$A_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$$
$$X = (x_1, x_2, \ldots, x_n)$$

Then $(*)$ is

$$\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ &\vdots \\ A_m \cdot X &= 0 \end{aligned} \qquad (**)$$

By subtracting a multiple of the first row and adding it to the rows below we can eliminate x_1 in row 2 through row m.

That is, (**) is equivalent to

$$\boxed{\begin{aligned} &A_1 \cdot X = 0 \\ &(A_2 - a_{21} a_{11}^{-1} A_1) \cdot X = 0 \\ &\vdots \\ &(A_m - a_{m1} a_{11}^{-1} A_1) \cdot X = 0 \end{aligned}}$$

The system

$$\boxed{\begin{aligned} &(A_2 - a_{21} a_{11}^{-1} A_1) \cdot X = 0 \\ &\vdots \\ &(A_m - a_{m1} a_{11}^{-1} A_1) \cdot X = 0 \end{aligned}} \quad (***)$$

is a system with $m-1$ equations and $n-1 > m-1$ unknowns.

Thus, by the induction hypothesis we can find a non-trivial solution $(x_2, x_3, \ldots, x_n)$ to (***)

Now using this solution $(x_2, \dots, x_n)$ to $(***)$ we can also solve $A_1 \cdot X = 0$

by setting

$$x_1 = -a_{11}^{-1}(a_{12}x_2 + \dots + a_{1n}x_n)$$

[Because the first eqn $A_1 \cdot X = 0$ is $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$]

Set $X = (x_1, x_2, \dots, x_n)$ from above.

We have $A_1 \cdot X = 0$.
And also, if $i \geq 2$ then

$$A_i \cdot X \underset{(***)}{=} a_{i1} a_{11}^{-1} \underbrace{A_1 \cdot X}_{0} = 0.$$

So we have solved
$A_1 \cdot X = 0$
$A_2 \cdot X = 0$
$\vdots$
$A_m \cdot X = 0$
with a non-trivial solution ▨

<u>Theorem:</u> Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_m \in V$ where $V = \text{span}(\{v_1, v_2, \ldots, v_m\})$. Let $w_1, w_2, \ldots, w_n \in V$. If $n > m$, then $w_1, w_2, \ldots, w_n$ are linearly dependent.

<u>proof:</u> Since $V = \text{span}(\{v_1, v_2, \ldots, v_m\})$ we have that

$$\begin{aligned} w_1 &= a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m \\ w_2 &= a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m \\ &\vdots \\ w_n &= a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m \end{aligned}$$

where $a_{ij} \in F$.

For any $c_1, c_2, \ldots, c_n \in F$ we have

$$c_1 w_1 + c_2 w_2 + \cdots + c_n w_n$$

$$= (c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n}) V_1$$

$$+ \cdots +$$

$$+ (c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn}) V_m$$

By the previous theorem, since $n > m$,

$$c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn} = 0$$

has a non-trivial solution $(c_1, c_2, \ldots, c_n) \neq \vec{0}$. This solution will yield $c_1 w_1 + c_2 \cdot w_2 + \cdots + c_n w_n = \vec{0}$ making $w_1, w_2, \ldots, w_n$ linearly dependent. ∎

Math 4570

9/16/20

(Continuation of HW 2 topics)

<u>Last time:</u> If $V = \text{span}(\{v_1, \ldots, v_m\})$ and $w_1, w_2, \ldots, w_n \in V$. If $n > m$ then $w_1, w_2, \ldots, w_n$ are linearly dependent.

<u>Corollary:</u> Let V be a vector space over a field F. Suppose $\beta_1 = \{v_1, v_2, \ldots, v_a\}$ and $\beta_2 = \{w_1, w_2, \ldots, w_b\}$ are both bases for V. Then, $a = b$.

<u>proof:</u> Since β_1 is a basis for V, $V = \text{span}(\{v_1, v_2, \ldots, v_a\})$. Since β_2 is a basis, $\beta_2 = \{w_1, w_2, \ldots, w_b\}$ are linearly independent. If $b > a$ then β_2 would be a linearly dependent set, which isn't the case. So, $b \leq a$.

Since β_2 is a basis for V,
$V = \text{span}(\{w_1, w_2, \dots, w_b\})$

Since β_1 is a basis, $\beta_1 = \{v_1, v_2, \dots, v_a\}$ is a linearly independent set.

If $a > b$, then from the previous thm β_1 would be linearly dependent.
So, $a \leq b$.

Thus since $b \leq a$ and $a \leq b$, we have $a = b$. $\blacksquare$

<u>Def:</u> Let V be a vector space over a field F. We say that V is <u>finite dimensional</u> if it has a basis consisting of a finite number of elements. If V has a basis with n elements then we say that V has <u>dimension n</u> and write $\dim(V) = n$.

A special case is the when $V = \{\vec{0}\}$. This vector space has no basis.
We define $V = \{\vec{0}\}$ to have dimension zero, ie $\dim(\{\vec{0}\}) = 0$

<u>Ex:</u> Let F be a field and $V = F^n$ where $n \geq 1$. F^n is a vector space over F.
Then $\dim(F^n) = n$

<u>proof:</u> Let v_i be the vector with 1 in the ith spot and 0's elsewhere.
That is,

$$v_1 = (1, 0, \ldots, 0)$$
$$v_2 = (0, 1, \ldots, 0)$$
$$\vdots$$
$$v_n = (0, 0, \ldots, 1)$$

Let $\beta = \{v_1, v_2, \ldots, v_n\}$.

If we show β is a basis for V over F, then $\dim(V) = n$.

<u>β spans F^n</u>:

Let $x \in F^n$.

Then $x = (f_1, f_2, \ldots, f_n)$ where $f_i \in F$.

Then

$$x = (f_1, 0, \ldots, 0) + (0, f_2, \ldots, 0) + \cdots + (0, 0, \ldots, f_n)$$

$$= f_1(1, 0, \ldots, 0) + f_2(0, 1, \ldots, 0) + \cdots + f_n(0, 0, \ldots, 1)$$

$$= f_1 v_1 + f_2 v_2 + \cdots + f_n v_n$$

So, $x \in \mathrm{span}(\beta)$.

β is a linearly independent set.

Suppose
$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}$$
where $c_1, c_2, \dots, c_n \in F$.

Then
$$c_1(1,0,\dots,0) + c_2(0,1,\dots,0) + \dots + c_n(0,0,\dots,1) = (0,0,\dots,0)$$

So,
$$(c_1, c_2, \dots, c_n) = (0, 0, \dots, 0).$$

Hence, $c_1 = 0$, $c_2 = 0$, $\dots$, $c_n = 0$.
So, β is a linearly independent set. $\blacksquare$

Ex: $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with $\dim(\mathbb{R}^n) = n$.

Ex: Let $F=\mathbb{R}$ or $F=\mathbb{C}$.

$$P_n(F) = \{ a_0 + a_1 x + \dots + a_n x^n \mid a_i \in F \}$$

One can show that

$$v_0 = 1$$
$$v_1 = x$$
$$v_2 = x^2$$
$$\vdots$$
$$v_n = x^n$$

is a basis for $P_n(F)$ over F.

So,

$$\dim(P_n(F)) = n+1$$

<u>Ex:</u> Let F be a field and $V = M_{m,n}(F)$ be the set of $m \times n$ matrices with entries from F.

$$M_{3,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{R} \right\}$$

basis for $M_{3,2}(\mathbb{R})$ over $\mathbb{R}$ is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $\dim(M_{3,2}(\mathbb{R})) = 6$

In general,

$$\dim(M_{m,n}(F)) = m \cdot n$$

Theorem: Let V be a vector space over a field F. Suppose $\dim(V) = n > 0$. Then:

(1) Let $v_1, v_2, \ldots, v_m \in V$.

(a) If $m > n$, then $v_1, v_2, \ldots, v_m$ are linearly dependent.

(b) If $m < n$, then $v_1, v_2, \ldots, v_m$ do not span V.

(c) If $m = n$ and $v_1, v_2, \ldots, v_m$ span V, then $v_1, v_2, \ldots, v_m$ is a linearly independent set and hence is a basis for V.

(d) If $m = n$ and $v_1, v_2, \ldots, v_m$ are linearly independent, then $v_1, v_2, \ldots, v_m$ span V and hence are a basis for V.

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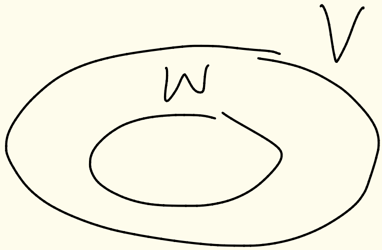
(2) Let W be a subspace of V. Then W is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, $\dim(W) = \dim(V)$ iff $V = W$.



<u>proof:</u> Let V be an n dimensional vector space over F. Let $v_1, v_2, \ldots, v_m \in V$.

(a) Suppose $m > n$.
Since V is spanned by n vectors, by the thm from Monday $v_1, v_2, \ldots, v_m$ are lin. dep. since $m > n$.

(b) Suppose $m<n$.
Let's show $v_1, v_2, \ldots, v_m$ do not span V.

Suppose they did, that is suppose
$V = \operatorname{span}(\{v_1, v_2, \ldots, v_m\})$.

Then any set of n vectors must be linearly dependent from Monday's theorem. [since $m<n$].

But V has a basis of size n, which consists of n lin. ind. vectors.

Contradiction.
So, $v_1, v_2, \ldots, v_m$ do not span V.

(c) Suppose $m=n$ and $\beta=\{v_1,v_2,\ldots,v_m\}$ spans V.

We want to show β is a lin. ind. set.

By a HW problem, there is a subset β' of β where β' is a basis for V.

HW 2

\# 7(b) Suppose $V\neq\{\vec{0}\}$ is spanned by some finite set S. Prove that some subset of S is a basis for V.

But since $\dim(V)=n$, the size of β' is n.

So, $\beta'=\beta$.

Thus, β is a linearly independent set.

(d) Suppose $m = n$ and $v_1, v_2, \ldots, v_m$ are linearly independent.

We want to show $V = \text{span}(\{v_1, v_2, \ldots, v_m\})$

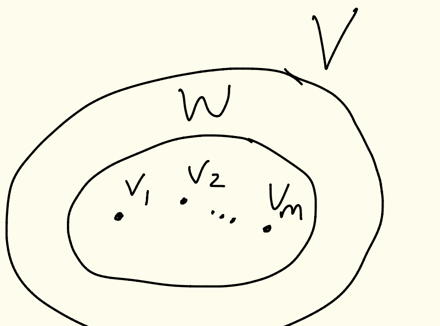
Let $W = \text{span}(\{v_1, v_2, \ldots, v_m\})$.

Let's show $W = V$.

We know $W \subseteq V$.

Let's show $V \subseteq W$.

Let $v \in V$.



Since $\dim(V) = n = m$ and $v_1, v_2, \ldots, v_m, v$ are $m+1 = n+1$ vectors, by part (a) $v_1, v_2, \ldots, v_m, v$ are linearly dependent.

So there exists $c_1, c_2, \ldots, c_m, c \in F$, not all zero, with

$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m + cv = \vec{0}$$

If $c = 0$, then $c_1 v_1 + \ldots + c_m v_m = \vec{0}$ with $c_1, c_2, \ldots, c_m$ not all zero. This can't happen because $v_1, v_2, \ldots, v_m$ are lin. ind.

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So, $c \neq 0$.

Thus,

$$v = -c^{-1}c_1 v_1 - c^{-1}c_2 v_2 - \dots - c^{-1}c_m v_m$$

So,

$$v \in \operatorname{span}(\{v_1, v_2, \dots, v_m\}) = W.$$

Thus, $V \subseteq W$ and $V = W$.

② Next time

$\vdots$

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continuation of
Theorem from last time

Let V be a vector space over a field F. Suppose $\dim(V) = n > 0$.

① (with 4 parts that we proved)

② Let W be a subspace of V. Then W is finite-dimensional and $\dim(W) \leq \dim(V)$.
Moreover, $\dim(W) = \dim(V)$ iff $W = V$.

proof of ②:

Let's first show that W is finite-dimensional and $\dim(W) \leq \dim(V)$.

If $W = \{\vec{0}\}$, then W is finite-dimensional with
$\dim(W) = 0 < n = \dim(V)$.

Suppose $\dim(W) \geq 1$.

Then there exists a non-zero vector $X_1 \in W$ where $\{X_1\}$ is a linearly independent set.

Continue adding vectors from W to this set such that at each stage k, the vectors $\{X_1, X_2, \ldots, X_k\}$ are linearly independent.

Since $W \subseteq V$ and $\dim(V) = n$, by part (a) of this theorem there must reach a stage $k_0 \leq n$ where $S_0 = \{X_1, X_2, \ldots, X_{k_0}\}$ is linearly independent but adding any new vector from W to S_0 will yield a linearly dependent set.

HW 2 7(a) Let S be a finite set of linearly independent vectors from V and let $x \in V$ where $x \notin S$. Then $S \cup \{x\}$ is linearly dependent iff $x \in \text{span}(S)$.

Let $x \in W$.

If $x \in S_0$, then $x \in \text{span}(S_0)$.

If $x \notin S_0$, then by the construction of S_0, we have $S_0 \cup \{x\}$ is linearly dependent.

So, by HW 2 7(a), in this case $x \in \text{span}(S_0)$.

Thus, $W = \text{span}(S_0)$.

And S_0 is a lin. ind. set, so is a basis for W.

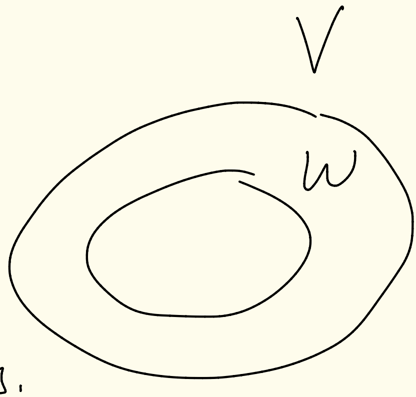
So, $\dim(W) = k_0 \leq n = \dim(V)$.

Now for this part:

$\dim(W) = \dim(V)$ iff $V = W$.

($\Leftarrow$) If $V = W$, then $\dim(W) = \dim(V)$.

($\Rightarrow$) Suppose now that $\dim(W) = \dim(V)$.



Then W has a basis with $n = \dim(W) = \dim(V)$ elements, call it $\beta = \{w_1, w_2, \ldots, w_n\}$.

By part 1(d) since β is a lin. ind. set of $n = \dim(V)$ vectors, β must span V and hence be a basis for V.

So, $W = \text{span}(\beta) = V$. $\blacksquare$

Linear Transformations

HW 3 material

<u>Def:</u> Let V and W be vector spaces over a field F. Let $T: V \to W$ be a function between them. We say that that T is a <u>linear transformation</u> if for all $v_1, v_2 \in V$ and $\alpha \in F$ we have that

$$\text{① } T(v_1 + v_2) = T(v_1) + T(v_2)$$

and

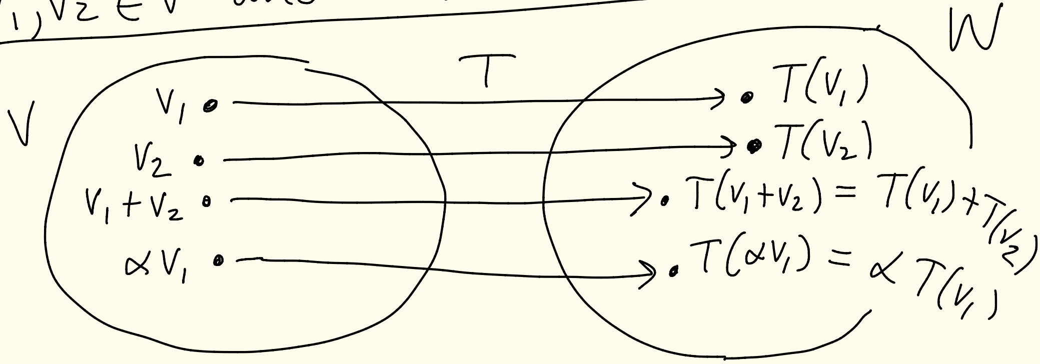
$$\text{② } T(\alpha v_1) = \alpha T(v_1)$$

Side note

Can combine ① & ② into one formula

$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$$

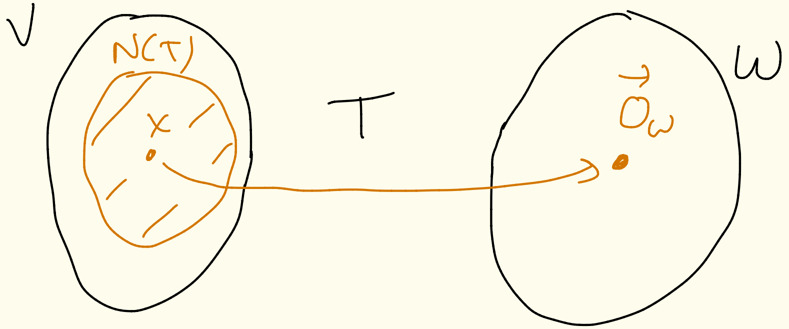
which must be true for all $v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in F$

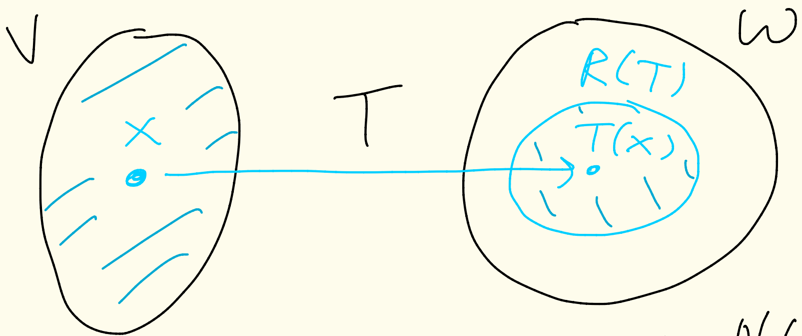


(def continued)

We define the <u>nullspace</u> (or <u>kernel</u>) of T to be $N(T) = \{ x \in V \mid T(x) = \vec{0}_W \}$ where $\vec{0}_W$ is the zero vector of W.

We define the <u>range</u> (or <u>image</u>) of T to be $R(T) = \{ T(x) \mid x \in V \}$





We will show later that $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

If $N(T)$ is finite-dimensional then we call the dimension of $N(T)$ the <u>nullity of T</u> and write $\text{nullity}(T) = \dim(N(T))$.

If $R(T)$ is finite-dimensional then we call the dimension of $R(T)$ the <u>rank of T</u> and write $\text{rank}(T) = \dim(R(T))$.

Ex: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x,y,z) = (x,y)$

Here $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$ and $F = \mathbb{R}$

$\boxed{T \text{ is linear}}$: Let $v_1, v_2 \in \mathbb{R}^3$.
So, $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$.
Let $\alpha \in \mathbb{R}$. Then:

(1) $$\begin{aligned} T(v_1 + v_2) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2, y_1 + y_2) \\ &= (x_1, y_1) + (x_2, y_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\ &= T(v_1) + T(v_2) \end{aligned}$$

(2) $$\begin{aligned} T(\alpha v_1) &= T(\alpha x_1, \alpha y_1, \alpha z_1) \\ &= (\alpha x_1, \alpha y_1) = \alpha (x_1, y_1) \\ &= \alpha T(x_1, y_1, z_1) = \alpha T(v_1). \end{aligned}$$

So, by (1) & (2) T is a linear transformation.

$\boxed{\text{Nullspace of } T}$:

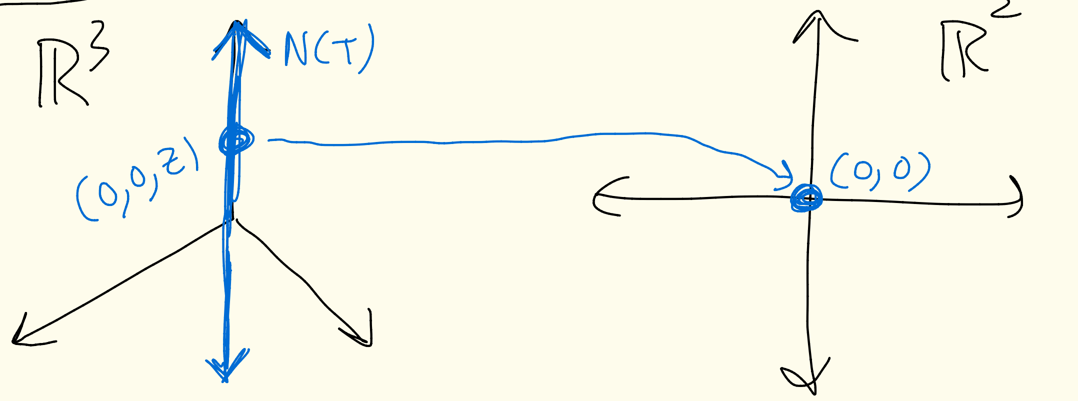
$$N(T) = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0)\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) = (0,0)\}$$
$$= \{(0,0,z) \mid z \in \mathbb{R}\}$$
$$= \{z(0,0,1) \mid z \in \mathbb{R}\} = \text{span}(\{(0,0,1)\})$$

Let $\beta = \{(0,0,1)\}$. Then, $\text{span}(\beta) = N(T)$.

Also, β is a lin. ind. set since it consists of one non-zero vector.

So, β is a basis for $N(T)$.

Thus, $\text{nullity}(T) = \dim(N(T)) = 1$.



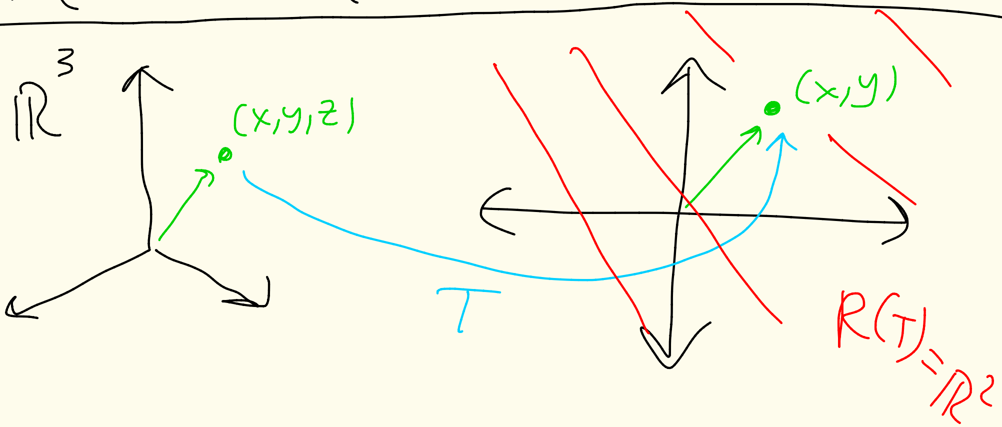
Range of T

$$R(T) = \{ T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3 \}$$
$$= \{ (x,y) \mid (x,y,z) \in \mathbb{R}^3 \}$$
$$= \{ (x,y) \mid x,y \in \mathbb{R} \} = \mathbb{R}^2$$

$$= \{ x(1,0) + y(0,1) \mid x,y \in \mathbb{R} \}$$
$$= \mathrm{span}(\{(1,0),(0,1)\})$$

side note

$$\mathrm{rank}(T) = \dim(R(T)) = \dim(\mathbb{R}^2) = 2$$



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Ex: Let

$$T : \underbrace{P_n(\mathbb{R})}_{\substack{\text{polynomials}\\ \text{of degree} \leq n\\ \text{coefficients in } \mathbb{R}}} \longrightarrow \underbrace{P_{n-1}(\mathbb{R})}_{\substack{\text{same but}\\ \text{degree} \leq n-1}}$$

$V = P_n(\mathbb{R})$
$W = P_{n-1}(\mathbb{R})$
$F = \mathbb{R}$

be defined by $T(f) = f'$ where f' is the usual derivative of f.

$\boxed{\text{T is linear:}}$

Let $f_1, f_2 \in P_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$.

Then

$$T(f_1 + f_2) = (f_1 + f_2)' = f_1' + f_2' = T(f_1) + T(f_2)$$

derivative property

and

$$T(\alpha f_1) = (\alpha f_1)' = \alpha f_1' = \alpha T(f_1)$$

So, T is a linear transformation.

$\boxed{\text{Nullspace of } T}$:

$N(T) =$

$= \{a_0 + a_1 x + \dots + a_n x^n \in P_n(\mathbb{R}) \mid T(a_0 + a_1 x + \dots + a_n x^n) = 0\}$

$= \{a_0 + a_1 x + \dots + a_n x^n \in P_n(\mathbb{R}) \mid a_1 + 2a_2 x + \dots + n a_n x^{n-1} = 0\}$

$= \{a_0 + a_1 x + \dots + a_n x^n \in P_n(\mathbb{R}) \mid a_1 = a_2 = \dots = a_n = 0\}$

$= \{a_0 \mid a_0 \in \mathbb{R}\}$ ← all the constant polynomials

$= \{a_0 \cdot 1 \mid a_0 \in \mathbb{R}\}$

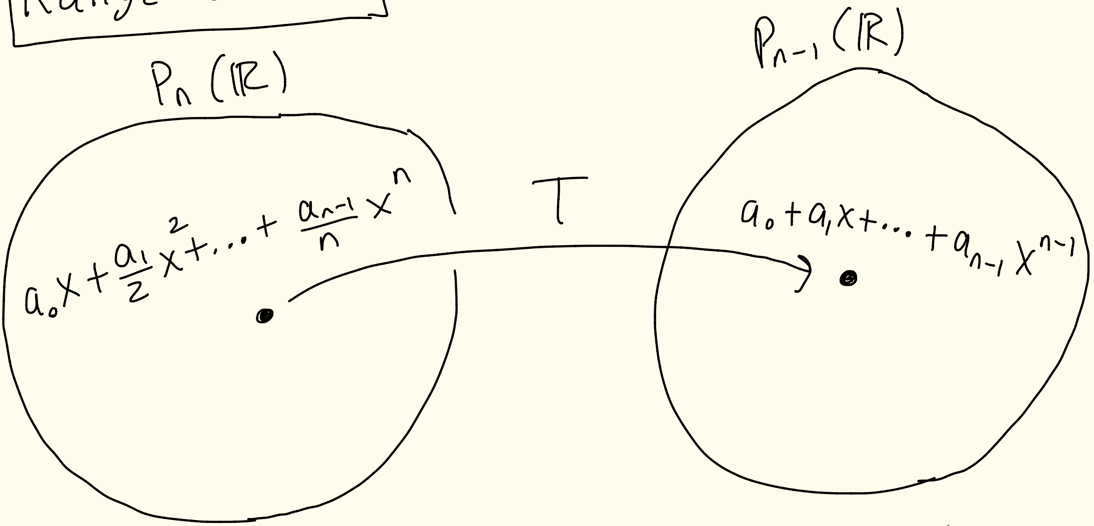
$= \text{span}(\{1\})$

So, a basis for $N(T)$ is $\beta = \{1\}$.

So, $\text{nullity}(T) = \dim(N(T)) = 1$

Recall a basis for $P_n(\mathbb{R})$ is $\{1, x, x^2, \dots, x^n\}$ so $\dim(P_n(\mathbb{R})) = n+1$

Range of T:



We claim that $\underbrace{R(T)}_{\text{range of } T} = P_{n-1}(\mathbb{R})$.

That is T is onto $P_{n-1}(\mathbb{R})$.

Let $a_0+a_1x+\dots+a_{n-1}x^{n-1} \in P_{n-1}(\mathbb{R})$.

Then $a_0x+\frac{a_1}{2}x^2+\dots+\frac{a_{n-1}}{n}x^n \in P_n(\mathbb{R})$

and $T\left(a_0x+\frac{a_1}{2}x^2+\dots+\frac{a_{n-1}}{n}x^n\right)$

$= a_0+a_1x+\dots+a_{n-1}x^{n-1}$.

So, T is onto.

$$\boxed{\begin{aligned} \operatorname{rank}(T) &= \dim(R(T)) \\ &= \dim(P_{n-1}(\mathbb{R})) \\ &= (n-1)+1 = n \end{aligned}}$$

Notice that

$$\dim\left(P_n(\mathbb{R})\right) = 1 + n$$

$\underbrace{\qquad\qquad}_{\text{dimension of domain}}$

$$= \dim(N(T)) + \dim(R(T))$$
$$= \text{nullity}(T) + \text{rank}(T)$$

Theorem: Let V and W be vector spaces over a field F. Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W respectively. Let $T: V \to W$ be a function.

Then:

① If T is a linear transformation then $T(\vec{0}_V) = \vec{0}_W$.

② T is a linear transformation iff $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in V$ and $\alpha, \beta \in F$.

③ T is a linear transformation iff $T(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) = a_1 T(x_1) + a_2 T(x_2) + \dots + a_n T(x_n)$ for all $x_1, x_2, \dots, x_n \in V$ and $a_1, a_2, \dots, a_n \in F$.

Matrix multiplication is a linear transformation. (Why?)

<u>Def:</u> Let F be a field.
Let A be an $m \times n$ matrix with coefficients from F. [ie, $A \in M_{m,n}(F)$]
We can construct a linear transformation $L_A : F^n \to F^m$ where $L_A(x) = Ax$ for any $x \in F^n$. Here Ax is matrix multiplication.
L_A is called <u>the left-multiplication by A transformation.</u>

<u>Note:</u> L_A is a linear transformation because if $\alpha, \beta \in F$ and $x, y \in F^n$ then

$$
\begin{aligned}
L_A(\alpha x + \beta y) &= A(\alpha x + \beta y) \\
&= A(\alpha x) + A(\beta y) \\
&= \alpha Ax + \beta Ay \\
&= \alpha L_A(x) + \beta L_A(y)
\end{aligned}
$$

properties of matrices

Ex: $F = \mathbb{C}$

$$A = \begin{pmatrix} 1 & i & 2i \\ 0 & -1 & -3i \end{pmatrix} \in M_{2,3}(\mathbb{C})$$

$$L_A : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$$

$$L_A\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & i & 2i \\ 0 & -1 & -3i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for example

$$L_A\begin{pmatrix} i \\ 0 \\ 2i \end{pmatrix} = \begin{pmatrix} 1 & i & 2i \\ 0 & -1 & -3i \end{pmatrix}\begin{pmatrix} i \\ 0 \\ 2i \end{pmatrix}$$

$$= \begin{pmatrix} (1)(i) + (i)(0) + \overbrace{(2i)(2i)}^{4i^2 = -4} \\ (0)(i) + (-1)(0) + \underbrace{(-3i)(2i)}_{-6i^2 = 6} \end{pmatrix}$$

$$i^2 = -1$$

$$= \begin{pmatrix} -4 + i \\ 6 \end{pmatrix}$$

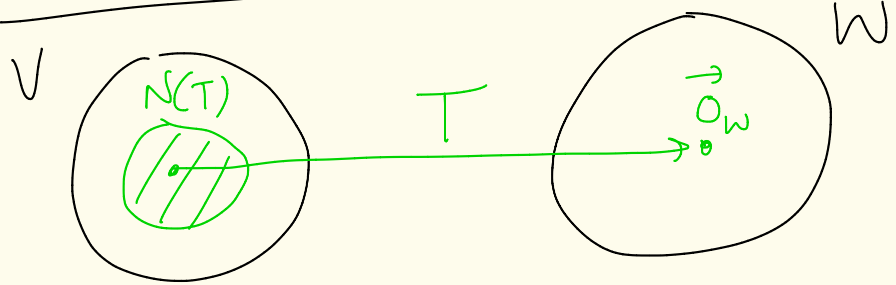
Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

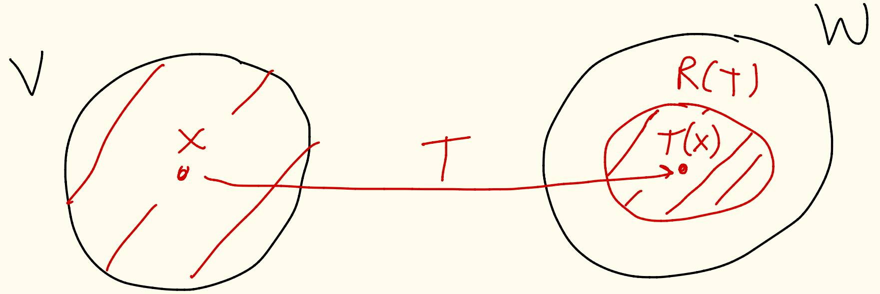
Then

① $N(T)$ is subspace of V

and

② $R(T)$ is a subspace of W





<u>proof:</u> Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W.

① Recall $N(T) = \{x \in V \mid T(x) = \vec{0}_W\}$.

(a) From a previous theorem (HW), $T(\vec{0}_V) = \vec{0}_W$. So, $\vec{0}_V \in N(T)$.

(b) Let $x, y \in N(T)$.
Then $T(x) = \vec{0}_W$ and $T(y) = \vec{0}_W$.

$$\text{So, } T(x+y) \underset{\boxed{T \text{ is linear}}}{=} T(x) + T(y) = \vec{0}_W + \vec{0}_W = \vec{0}_W.$$

Since $T(x+y) = \vec{0}_W$ we have $x + y \in N(T)$.

(c) Let $x \in N(T)$ and $\alpha \in F$.
Then $T(x) = \vec{0}_W$ $\boxed{\text{since } x \in N(T)}$

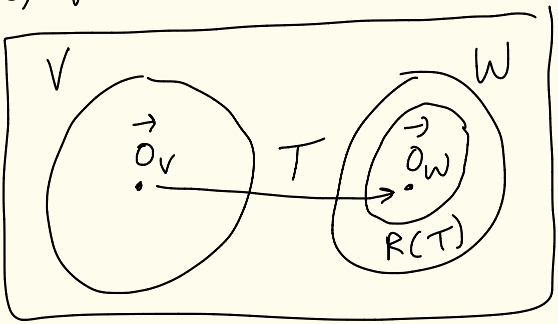
Then $T(\alpha x) \underset{\boxed{T \text{ is linear}}}{=} \alpha T(x) = \alpha \vec{0}_W = \vec{0}_W$.
So, $\alpha x \in N(T)$.

By (a), (b), (c), $N(T)$ is a subspace of V.

② Recall that

$$R(T) = \{ T(x) \mid x \in V \}.$$

(a) We have that $\vec{0}_W = \underbrace{T(\vec{0}_V)}_{\text{something in } R(T)} \in R(T)$.



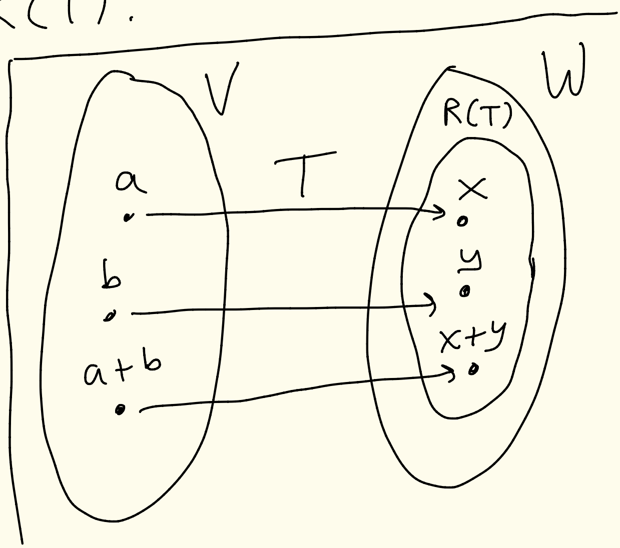
(b) Let $x, y \in R(T)$.

So there exist $a, b \in V$ where $T(a) = x$ and $T(b) = y$

Because T is linear

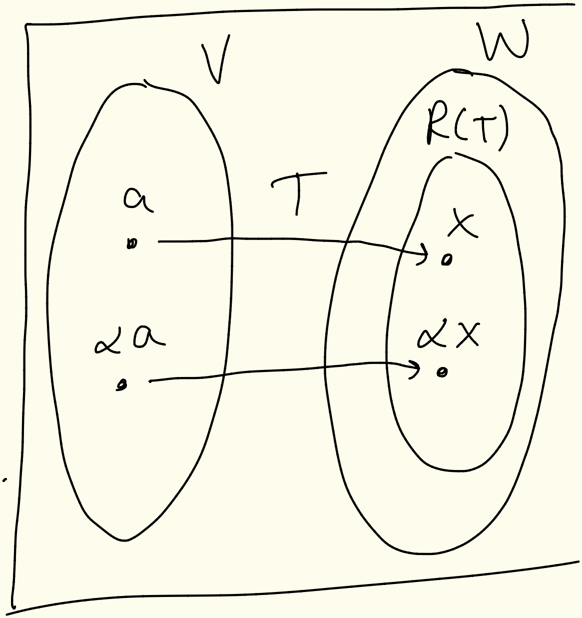
$$x + y = T(a) + T(b) = T(a+b) \in R(T)$$

So, $x + y \in R(T)$.



(c) Let $x \in R(T)$ and $\alpha \in F$.

Since $x \in R(T)$ there exists $a \in V$ with $T(a) = x$.



Since T is linear

$$\alpha x = \alpha T(a) = T(\alpha a) \in R(T).$$

So, $\alpha x \in R(T)$.

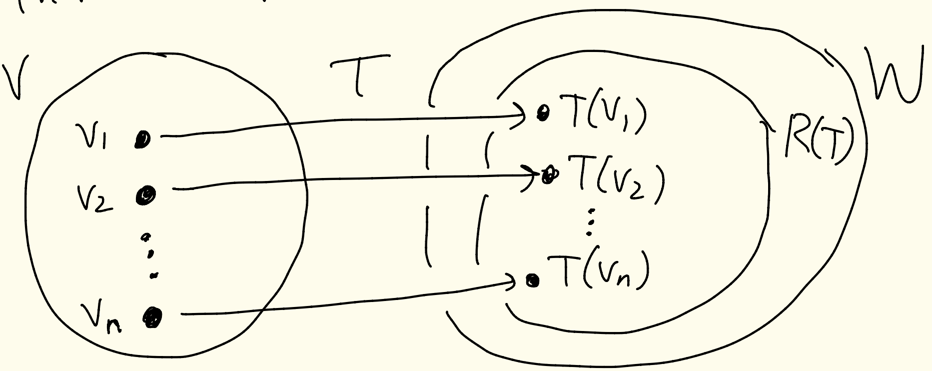
By parts (a), (b), (c), $R(T)$ is a subspace of W. $\blacksquare$

Math 4570

9/28/20

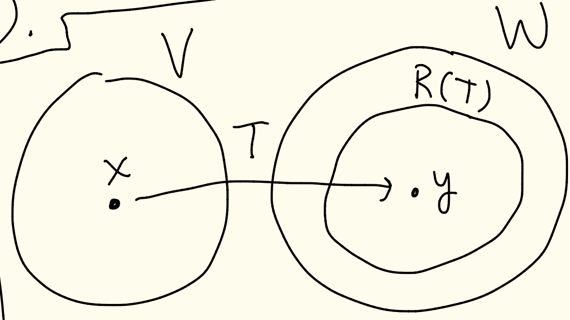
Today we will prove the rank/nullity theorem. First we need some tools.

Lemma: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation. If $v_1, v_2, \ldots, v_n \in V$ and $V = \text{span}(\{v_1, v_2, \ldots, v_n\})$ then $R(T) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$.



proof: (HW problem).

Let $y \in R(T)$. By def of $R(T)$, there exists $x \in V$ with $T(x) = y$.



pg 2

Since $V = \operatorname{span}(\{v_1, v_2, \ldots, v_n\})$ and $x \in V$ we can write $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Applying T we get

$$y = T(x) = T(\alpha_1 v_1 + \alpha_2 v_1 + \cdots + \alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n).$$

T is linear

So, $y \in \operatorname{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$.

So, $R(T) = \operatorname{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$.

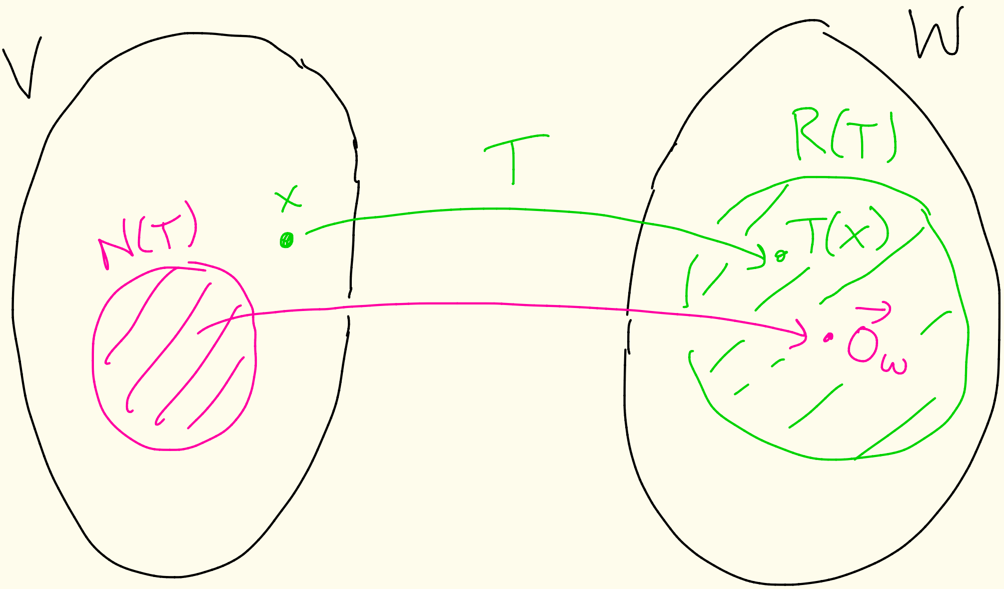
$\blacksquare$

Rank-Nullity Theorem

Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

If V is finite-dimensional, then

$$\dim(V) = \underbrace{\text{nullity}(T)}_{\dim(N(T))} + \underbrace{\text{rank}(T)}_{\dim(R(T))}$$



<u>proof:</u> Let $n = \dim(V)$.

Since $N(T)$ is a subspace of V, we must have that $N(T)$ is also finite dimensional.

Let $k = \dim(N(T))$.

Then $k \leq n$.

thm from class

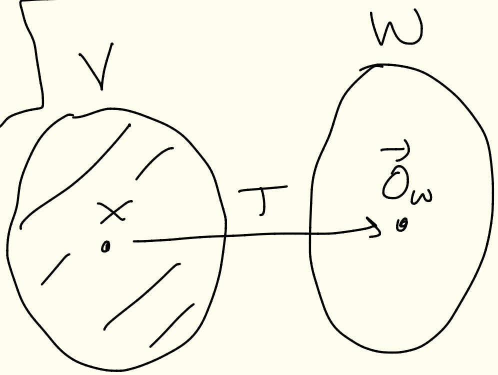
Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for $N(T)$.

Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W.

<u>case 1: Suppose $\dim(R(T)) = 0$</u>

Then $R(T) = \{\vec{0}_W\}$.

Then every $x \in V$ satisfies $T(x) = \vec{0}_W$.



So, $N(T) = V$.

So, $\dim(V) = \dim(N(T)) + 0$

$= \dim(N(T)) + \dim(R(T))$.

Case 2: Suppose $\dim(R(T)) \geq 1$

So there exists some non-zero vector in $R(T)$ and thus $N(T) \neq V$.

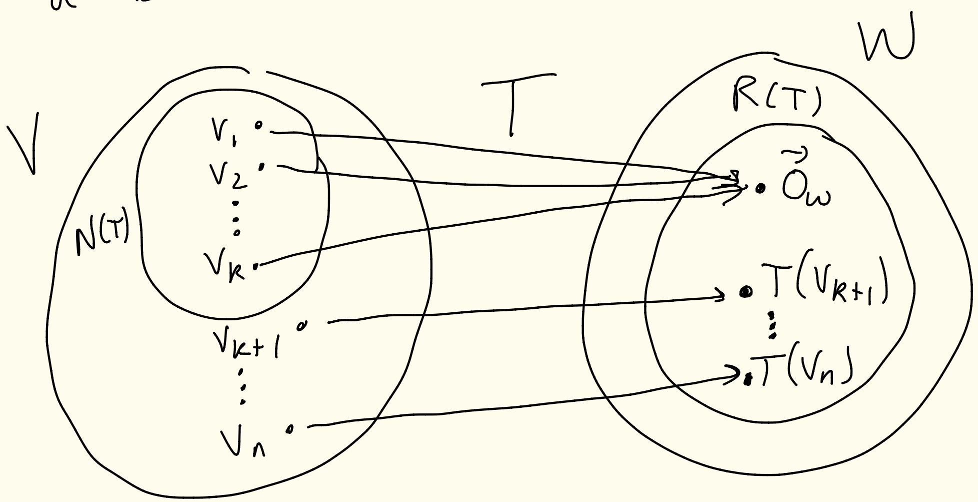
By HW 2 #9 we can extend the basis from $N(T)$ to all of V.

That is there exist
$v_{k+1}, v_{k+2}, \ldots, v_n \in V - N(T)$

where

$$\beta = \{ \underbrace{v_1, v_2, \ldots, v_k}_{\text{in } N(T)}, \underbrace{v_{k+1}, v_{k+2}, \ldots, v_n}_{\text{not in } N(T)} \}$$

is a basis for V.



We will show that

$$\beta' = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$$

is a basis for $R(T)$.

Once that's done we've proven the thm since then

$$\dim(V) = n = k + \underbrace{(n-k)}_{\text{size of } \beta'}$$
$$= \dim(N(T)) + \dim(R(T))$$

By the previous theorem, since β spans V, we know that

$$R(\mathbf{T}) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_k), T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\})$$

$$= \text{span}(\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\})$$

since $T(v_1) = T(v_2) = \cdots = T(v_k) = \vec{0}_W$

So, β' spans $R(T)$.

We just need to show β' is a linearly independent set.

Suppose

$$\alpha_{k+1} T(v_{k+1}) + \alpha_{k+2} T(v_{k+2}) + \cdots + \alpha_n T(v_n) = \vec{0}_W$$

where $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n \in F$.

Since T is linear we get

$$T(\alpha_{k+1} v_{k+1} + \alpha_{k+2} v_{k+2} + \cdots + \alpha_n v_n) = \vec{0}_W$$

So, $\alpha_{k+1} v_{k+1} + \alpha_{k+2} v_{k+2} + \cdots + \alpha_n v_n \in N(T)$.

Since $N(T) = \text{span}(\{v_1, v_2, \ldots, v_k\})$ we have

$$\alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$.

So,

$$-\alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_k v_k + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n = \vec{0}_V$$

But $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ is a basis for V, so $0 = (-\alpha_1) = (-\alpha_2) = \cdots = (-\alpha_k)$

$= \alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n$

So, $\beta' = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ pg 8

is a lin. ind. set.

Thus, β' is a basis for $R(T)$.

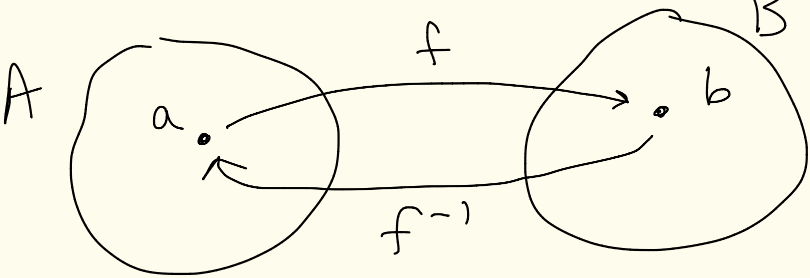
So, the thm is proved. $\blacksquare$

<u>Recall:</u>

Suppose $f: A \to B$ is 1-1 and onto where A and B are sets.

Then $f^{-1}: B \to A$ is defined by

$f^{-1}(b) = a$ iff $f(a) = b$

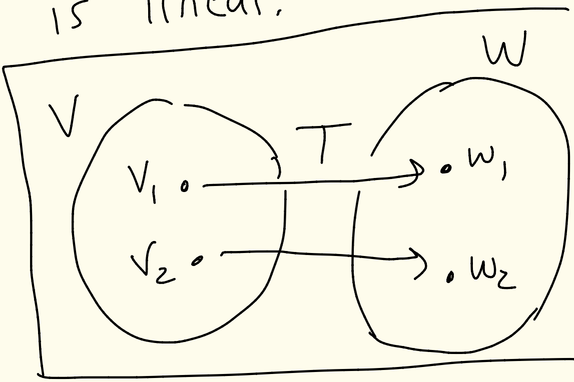


<u>Thm:</u> Let V and W be vector spaces over a field F. Let $T: V \to W$ be a 1-1 and onto linear transformation.
Then $T^{-1}: W \to V$ is also a linear transformation.

<u>proof:</u> Let $\alpha_1, \alpha_2 \in F$ and $w_1, w_2 \in W$. We will show

$$T^{-1}(\alpha_1 w_1 + \alpha_2 w_2) = \alpha_1 T^{-1}(w_1) + \alpha_2 T(w_2)$$

This will show T^{-1} is linear.



Since T is onto there exists $v_1, v_2 \in V$ where $T(v_1) = w_1$ and $T(v_2) = w_2$. By def of T^{-1} this means $T^{-1}(w_1) = v_1$ and $T^{-1}(w_2) = v_2$

Thus,

$$T^{-1}(\alpha_1 w_1 + \alpha_2 w_2)$$

$$= T^{-1}(\alpha_1 T(v_1) + \alpha_2 T(v_2))$$

$$= T^{-1}(T(\alpha_1 v_1 + \alpha_2 v_2))$$

T is linear

$$= \alpha_1 v_1 + \alpha_2 v_2$$

$\boxed{(T^{-1} \circ T)(x) = x \text{ for all } x \in V}$

$$= \alpha_1 T^{-1}(w_1) + \alpha_2 T^{-1}(w_2)$$

So, T^{-1} is linear. $\blacksquare$

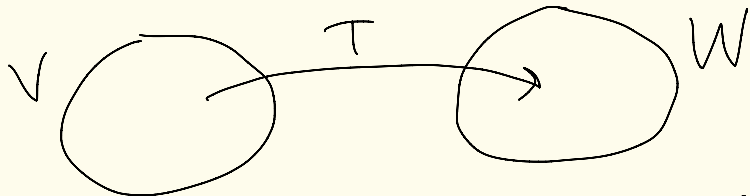
Math 4570

9/30/20

Test 1 will cover
HW 1 and HW 2

Def: Let V and W be vector spaces over a field F,

① An isomorphism between V and W is a linear transformation $T: V \to W$ that is 1-1 and onto.



② We say that V and W are isomorphic, and write $V \cong W$, if there exists an isomorphism T between them.

pg 2

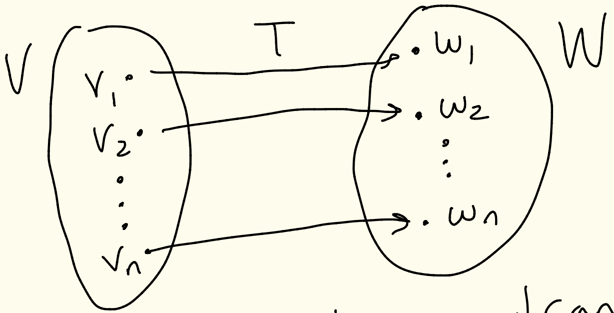
This def is well-defined since if $T: V \to W$ is an isomorphism, then by Monday's thereom $T^{-1}: W \to V$ is also an isomorphism.

That is, if $V \cong W$ by $T: V \to W$ then $W \cong V$ by $T^{-1}: W \to V$.

Theorem: Let V and W be vector spaces over a field F. Suppose that V is finite-dimensional and $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

$\boxed{\text{part 1}}$ Let $w_1, w_2, \ldots, w_n \in W$.

(1) There exists a unique linear transformation $T: V \to W$ where $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$



this unique linear transformation is given by the formula

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \quad (*)$$

(2) T given above is an isomorphism iff $\beta' = \{w_1, w_2, \ldots, w_n\}$ is a basis for W.

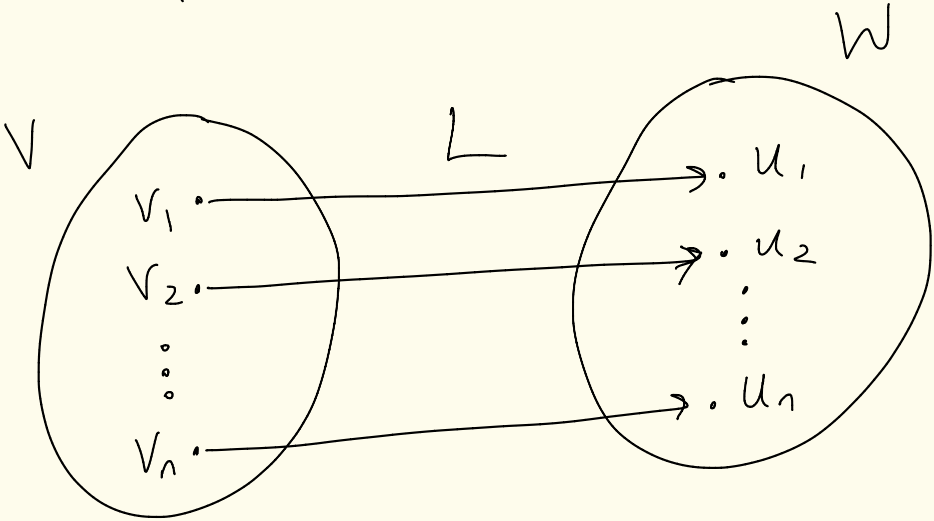
part 2

All linear transformations between V and W are constructed as in ① above. That is, if $L : V \to W$ is a linear transformation, set

$$u_i = L(v_i) \quad \text{for } i = 1, 2, \ldots, n$$

and then the formula for L is

$$L(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$
$$= c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$



proof: $\boxed{\text{part 1}}$ pg 5

① Let T be defined by $(*)$.

That is,

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$$

for any $c_i \in F$.

Let's show T is a linear transformation and $T(v_i) = w_i$ for all i.

<u>Why is T linear?</u>

Let $x, y \in V$ and $\alpha, \delta \in F$.

Since β is a basis for V, we can write $x = e_1 v_1 + \dots + e_n v_n$ and $y = d_1 v_1 + \dots + d_n v_n$ where $e_i, d_i \in F$. Then,

$$T(\alpha x + \delta y)$$

$$= T\big(\alpha(e_1 v_1 + \dots + e_n v_n) + \delta(d_1 v_1 + \dots + d_n v_n)\big)$$

$$= T\big((\alpha e_1 + \delta d_1) v_1 + \dots + (\alpha e_n + \delta d_n) v_n\big) =$$

$=$

pg 6

$= T\left((\alpha e_1 + \delta d_1)v_1 + \cdots + (\alpha e_n + \delta d_n)v_n\right)$

$\overset{(*)}{=} (\alpha e_1 + \delta d_1)w_1 + \cdots + (\alpha e_n + \delta d_n)w_n$

$= \alpha e_1 w_1 + \cdots + \alpha e_n w_n$
$\quad + \delta d_1 w_1 + \cdots + \delta d_n w_n$

$= \alpha(e_1 w_1 + \cdots + e_n w_n)$
$\quad + \delta(d_1 w_1 + \cdots + d_n w_n)$

$\overset{(*)}{=} \alpha T(e_1 v_1 + \cdots + e_n v_n)$
$\quad + \delta T(d_1 v_1 + \cdots + d_n v_n)$

$= \alpha T(x) + \delta T(y).$

So, T is linear.

Also,

$T(v_1) = T(1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n) = 1 \cdot w_1 = w_1$

$\vdots \qquad\qquad \vdots$

$T(v_n) = T(0 \cdot v_1 + 0 \cdot v_2 + \cdots + 1 \cdot v_n) = 1 \cdot w_n = w_n$

So, $T(v_i) = w_i$ for all i.

Why is T unique?

Suppose $S: V \to W$ is another linear transformation with $S(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

Let $x \in V$.

Then, since β is a basis for V,

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

And,

$$
\begin{aligned}
S(x) &= S(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\
&\overset{\text{S is linear}}{=} c_1 S(v_1) + c_2 S(v_2) + \cdots + c_n S(v_n) \\
&\overset{S(v_i) = w_i}{=} c_1 w_1 + c_2 v_2 + \cdots + c_n w_n \\
&= T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\
&\overset{\text{def of } T}{=} T(x)
\end{aligned}
$$

So, $S = T$ on V.

So, T is the unique linear transf. with $T(v_i) = w_i \ \forall i$

② $\boxed{\text{T defined by } (*) \text{ is an isomorphism iff } \beta' = \{w_1, w_2, \dots, w_n\} \text{ is a basis for } W.}$

($\Leftarrow$) Suppose β' is a basis for W. Let's show that T defined by $(*)$ is 1-1 and onto, and hence an isomorphism

$\boxed{\text{1-1}}$: Suppose $T(x) = T(y)$ for some $x, y \in V$.

Since β is a basis for V, $x = c_1 v_1 + \dots + c_n v_n$ and $y = d_1 v_1 + \dots + d_n v_n$ for $c_i, d_i \in F$.

Since $T(x) = T(y)$, by def of T, we have

$$\underbrace{c_1 w_1 + \dots + c_n w_n}_{T(x)} = \underbrace{d_1 w_1 + \dots + d_n w_n}_{T(y)}$$

So, $(c_1 - d_1) w_1 + \dots + (c_n - d_n) w_n = \vec{0}$

By assumption, β' is a lin. ind. set, so $0 = c_1 - d_1 = c_2 - d_2 = \dots = c_n - d_n$

So, $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$
and hence

$$x = c_1 v_1 + \cdots + c_n v_n = d_1 v_1 + \cdots + d_n v_n = y.$$

$\boxed{\text{onto}}$: We need to show $R(T) = W$.
By a previous thm, since $\beta = \{v_1, v_2, \ldots, v_n\}$
spans V, we know $R(T) = \text{span}(\{T(v_1), \ldots, T(v_n)\})$.
So,

$$R(T) = \text{span}(\{T(v_1), \ldots, T(v_n)\})$$
$$= \text{span}(\{w_1, \ldots, w_n\})$$
$$= W.$$

we are assuming $\beta' = \{w_1, \ldots, w_n\}$ is a basis for W

So, T is onto W.

Thus, T is an isomorphism.

(⇒) Now suppose T is an isomorphism, ie 1-1 and onto. Let's show β' is a basis for W.

Since T is onto, $R(T) = W$.

Therefore,

$$W = R(T) = \text{span}(\{T(v_1), \ldots, T(v_n)\})$$
$$= \text{span}(\{w_1, \ldots, w_n\})$$

So, β' spans W.

Is β' a lin. ind. set?

Suppose

$$d_1 w_1 + \ldots + d_n w_n = \vec{0}_W$$

where $d_i \in F$.

Since T is 1-1 and onto, T^{-1} exists and is linear (from Monday) and $T^{-1}(w_i) = v_i$ for $i = 1, \ldots, n$.

Since T^{-1} is linear, $T^{-1}(\vec{O}_w)=\vec{O}_v$.

So,

$$\vec{O}_v = T^{-1}(\vec{O}_w) = T^{-1}(d_1 w_1 + \dots + d_n w_n)$$

$$= d_1 T^{-1}(w_1) + \dots + d_n T^{-1}(w_n)$$

$$= d_1 v_1 + \dots + d_n v_n$$

Since $\beta = \{v_1, \dots, v_n\}$ is a basis and $\vec{O}_v = d_1 v_1 + \dots + d_n v_n$

we get $d_1 = d_2 = \dots = d_n = 0$.

Thus, β' is a lin. ind. set.

since if $d_1 w_1 + \dots + d_n w_n = \vec{O}_w$

then $d_1 = d_2 = \dots = d_n = 0$.

So, β' is a basis for W.

part 2

Suppose L is a linear transformation and $u_i = L(v_i)$ for $i = 1, 2, \ldots, n$.

Then,

$$L(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 L(v_1) + \cdots + c_n L(v_n)$$

$$= c_1 u_1 + \cdots + c_n u_n.$$

L is linear

$\blacksquare$

Math 4570

10/5/20

Test 1 on 10/21
No class that day.

Test 1 covers HW 1 & HW 2 material.

It will be something like where I email / give you the test on 10/21 in the morning. You'll have ≈ 24 hours. You pick a 2 hr window in that 24 hrs to take the test.

Ex: $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$

Let's construct a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ using the method from the theorem from Weds.

Step 1: Pick a basis for $\mathbb{R}^3$.

Let's use the standard basis

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \{ v_1, v_2, v_3 \}$$

(in the thm)

Step 2: Decide where the basis elements go.

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_1$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = w_2$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = w_3$$

pg 3

This will now determine a formula for T.

Let $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$.

Then, to make T linear we must define its formula as follows:

$$T(x) = T\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= T\left(a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

formula from thm. Need so T will be linear

$$= aT\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + cT\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

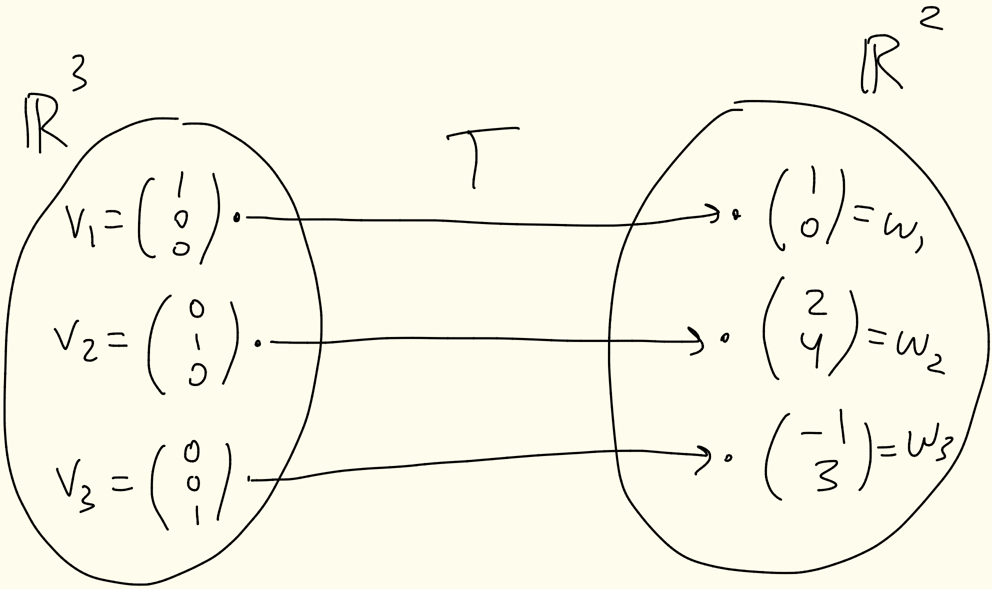
$$= a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 4 \end{pmatrix} + c\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} a + 2b - c \\ 4b + 3c \end{pmatrix}.$$

So, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+2b-c \\ 4b+3c \end{pmatrix}$ is a linear transformation.

T is an isomorphism iff $\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$. from Thm from Weds

$\dim(\mathbb{R}^2) = 2$, so every basis of $\mathbb{R}^2$ has 2 elements.
But β' above has 3 elements,
so β' is not a basis for $\mathbb{R}^2$.
So T is NOT an isomorphism.



Another way to show T is not an isomorphism is to show that T is not 1-1 using the formula for T

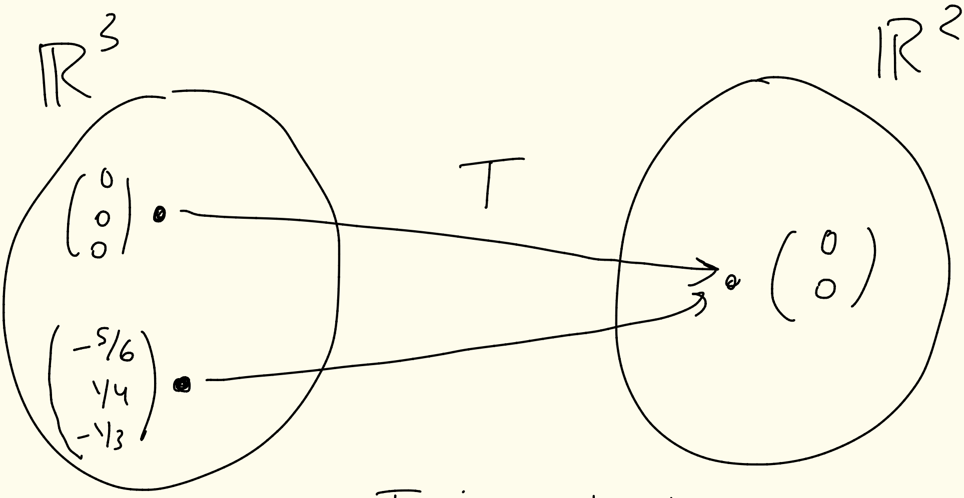
$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+2b-c \\ 4b+3c \end{pmatrix}$$

isomorphism means

1. linear transf.
2. 1-1
3. onto

T is not 1-1 because

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} -5/6 \\ 1/4 \\ -1/3 \end{pmatrix}$$



T is not 1-1

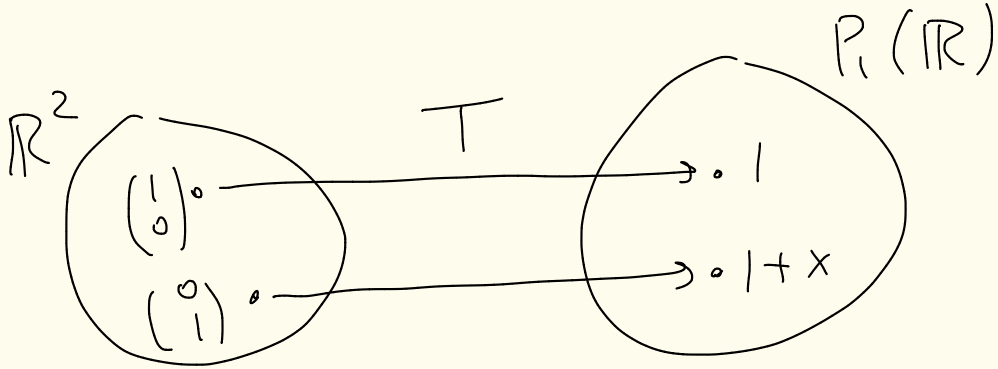
<u>Ex:</u> $V = \mathbb{R}^2$

$W = P_1(\mathbb{R}) = \{a + bx \mid a, b \in \mathbb{R}\}$

Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis for $\mathbb{R}^2$

Let $\beta' = \{1, 1+x\} \subseteq P_1(\mathbb{R})$.

Define a linear transformation $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ where $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 + x$



Then to define T for all of $\mathbb{R}^2$ to make it linear we must have:

$$T\begin{pmatrix} a \\ b \end{pmatrix} = T\left(a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= a \cdot 1 + b \cdot (1 + x)$$

$$= (a+b) + bx$$

So, $T\begin{pmatrix} a \\ b \end{pmatrix} = (a+b) + bx$

is a linear transformation from $\mathbb{R}^2$ to $P_1(\mathbb{R})$.

Is T an isomorphism?

T is an isomorphism iff $\beta' = \{1, 1+x\}$ is a basis for $\mathbb{R}^2$.

Is β' a linearly independent set?

Suppose

$$c_1 \cdot 1 + c_2 \cdot (1+x) = \vec{0}$$

$0 + 0x$

where $c_1, c_2 \in \mathbb{R}$.

We have

$$\underbrace{(c_1 + c_2)} \cdot 1 + \underbrace{c_2} \cdot x = 0 + 0x.$$

So, $\boxed{c_1 + c_2 = 0}$ and $\boxed{c_2 = 0.}$

So, $c_2 = 0$ and $c_1 = -c_2 = 0$.

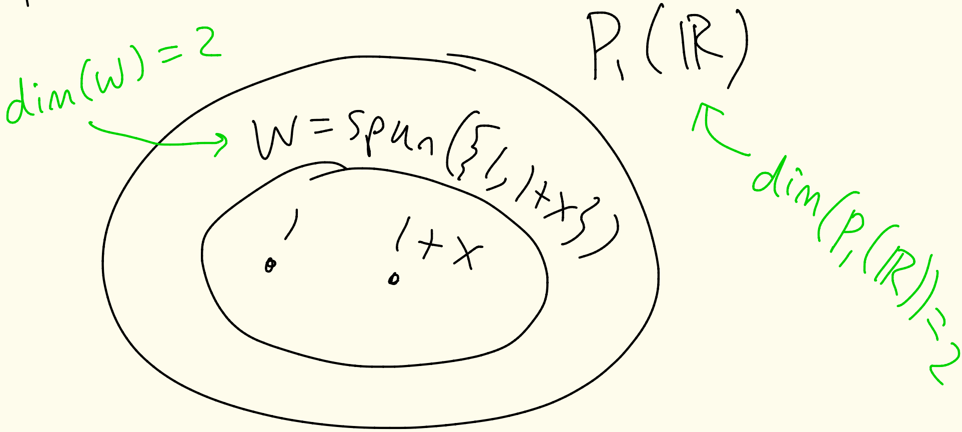
Thus, β' is a lin. ind. set.

Since $\beta' = \{1, 1+x\}$ consists of 2 linearly independent vectors, $W = \text{span}(\beta')$ has dimension 2.

Since $W \subseteq P_1(\mathbb{R})$ and $\dim(P_1(\mathbb{R})) = 2$ we have $W = P_1(\mathbb{R})$.

So, $W = \text{span}(\beta') = P_1(\mathbb{R})$.

So, β' is a basis for $P_1(\mathbb{R})$.



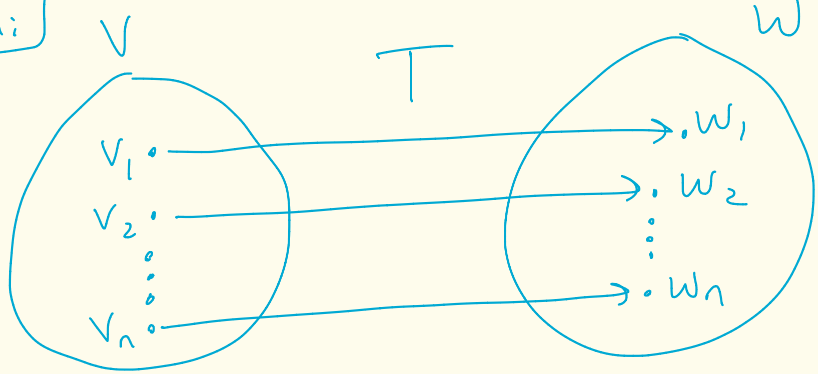
So, β' is a basis for $P_1(\mathbb{R})$ and T is an isomorphism.

Theorem: Let V and W be finite-dimensional vector spaces over a field F. We have that V and W are isomorphic iff $\dim(V) = \dim(W)$.

proof:

($\Leftarrow$) Suppose $\dim(V) = \dim(W) = n$. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for V and $\beta' = \{w_1, w_2, \ldots, w_n\}$ be a basis for W.

Idea:



Then, the function $T: V \to W$ given by

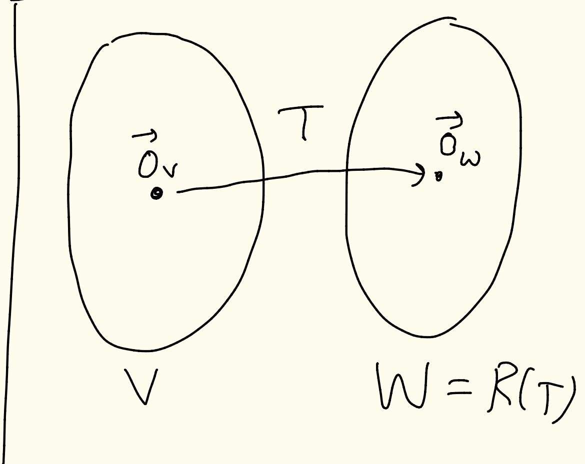
$$T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) = c_1 w_1 + c_2 w_2 + \dots + c_n w_n$$

will be an isomorphism by thm from Weds since $\beta' = \{w_1, w_2, \dots, w_n\}$ is a basis for W.

($\Longrightarrow$) Suppose V and W are isomorphic. This means that there exists a linear transformation $T: V \to W$ that is 1-1 and onto.

Since T is 1-1, from HW, $N(T) = \{\vec{0}_V\}$.

Since T is onto, $R(T) = W$.



By the rank/nullity thm,

$$\dim(V) = \text{nullity}(T) + \text{rank}(T)$$
$$= \dim(N(T)) + \dim(R(T))$$
$$= \underbrace{\dim(\{\vec{0}_V\})}_{0} + \dim(W)$$
$$= \dim(W).$$

So, $\dim(V) = \dim(W)$. $\blacksquare$

Corollary: Let V be a finite-dimensional vector space over a field F. Let $\dim(V) = n$.

Then, $V \cong F^n$.

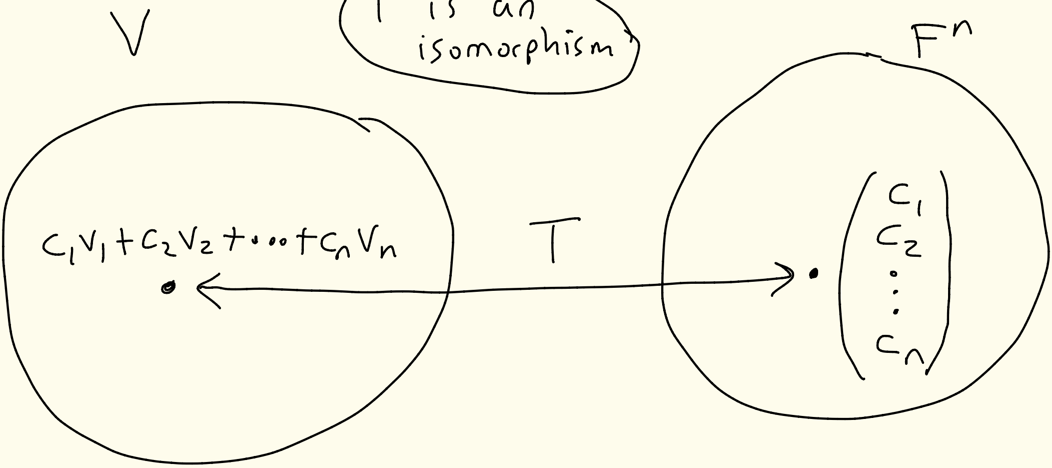
proof: $\dim(V) = n = \dim(F^n)$.

So, by the previous thm, $V \cong F^n$. $\blacksquare$

$\boxed{\text{Idea:}}$ $\boxed{V \text{ has basis } v_1, v_2, \ldots, v_n}$

$$\boxed{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}$$

T is an isomorphism



Recall:

A linear transformation $T: V \to W$ is called an isomorphism if T is 1-1 and onto.

Some people use the term invertible instead of isomorphism.

Math 4570

10/7/20

Test 1 (HW 1, HW 2)

Good calculation type questions

- checking if a set is a subspace (proof)
- finding the dimension and basis of a subspace or vector space
- checking if a set is lin. ind.
- checking if a set spans a space
- checking if a set is a basis
- checking if a vector is in the span of a set.
- checking if V is a vector space
- F^n, $P_n(F)$, $M_{m,n}(F)$

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proofs - practice HW

The Matrix of a Linear Transformation (HW 4 material)

Def: Let V be a finite-dimensional vector space over a field F. Suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. We write $\beta = [v_1, v_2, \ldots, v_n]$ to mean that β is an <u>ordered basis</u> for V, that is the order of the vectors in β is given and fixed.

Def: Suppose V is a finite-dimensional vector space over a field F with an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$. Let $x \in V$. Write $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. We write

$$[x]_\beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

and call $[x]_\beta$ the <u>coordinates of x with respect to β</u>. (or coordinate vector with respect to β)

Ex: $V=\mathbb{R}^2$, $F=\mathbb{R}$

$\beta = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$

you can check this is a basis for $\mathbb{R}^2$. Show $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are lin. ind. and then since there are 2 of them and $\dim(\mathbb{R}^2)=2$ they must span $\mathbb{R}^2$

Let $x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Let's find $[x]_\beta$.

Solve: $\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\boxed{\begin{aligned} 5 &= \alpha_1 - \alpha_2 \\ 4 &= 2\alpha_1 + \alpha_2 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right) \xrightarrow{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 3 & -6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \end{array}\right)$$

$$\boxed{\begin{aligned} \alpha_1 - \alpha_2 &= 5 \\ \alpha_2 &= -2 \end{aligned}}$$

$\alpha_1 = 3, \ \alpha_2 = -2$

2550 page has notes & worked problems on solving linear systems

So,

$$\begin{pmatrix}5\\4\end{pmatrix} = 3\begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}-1\\1\end{pmatrix}$$

Thus, $[x]_\beta = \left[\begin{pmatrix}5\\4\end{pmatrix}\right]_\beta = \begin{pmatrix}3\\-2\end{pmatrix}$

Let $\beta' = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$.

standard basis for $\mathbb{R}^2$

Then, $\begin{pmatrix}5\\4\end{pmatrix} = 5\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix}$

So, $[x]_{\beta'} = \begin{pmatrix}5\\4\end{pmatrix}$.

Ex:

$V = P_2(\mathbb{R}) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$

$F = \mathbb{R}$

$\beta = [1, 1+x, 1+x+x^2]$

you can show this is a basis. Show the vectors are lin. ind. Then since are 3 of them and $P_2(\mathbb{R})$ has dimension 3 they must span $P_2(\mathbb{R})$.

Let

$v = 2 - x + 3x^2.$

Find $[v]_\beta$.

$$2 - x + 3x^2 = \alpha_1 \cdot 1 + \alpha_2(1+x) + \alpha_3(1+x+x^2)$$

$$2 - x + 3x^2 = (\alpha_1 + \alpha_2 + \alpha_3) \cdot 1 + (\alpha_2 + \alpha_3)x + \alpha_3 x^2$$

So need to solve:

$$\boxed{\begin{aligned} 2 &= \alpha_1 + \alpha_2 + \alpha_3 \\ -1 &= \alpha_2 + \alpha_3 \\ 3 &= \alpha_3 \end{aligned}}$$

$$\boxed{\begin{aligned} \alpha_1 &= 3 \\ \alpha_2 &= -4 \\ \alpha_3 &= 3 \end{aligned}}$$

So,

$$[v]_\beta = [2 - x + 3x^2]_\beta = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$$

<u>Def:</u> Let $L : V \to W$ be a linear transformation between two finite-dimensional vector spaces V and W both over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V and γ be an ordered basis for W. The matrix

$$[L]_\beta^\gamma = \left(\underbrace{[L(v_1)]_\gamma}_{\text{column vector}} \;\middle|\; \underbrace{[L(v_2)]_\gamma}_{\text{column vector}} \;\middle|\; \cdots \;\middle|\; \underbrace{[L(v_n)]_\gamma}_{\text{column vector}} \right)$$

is called the <u>matrix for L with respect to β and γ</u>.

If $V = W$ and $\beta = \gamma$, then we write $[L]_\beta$ instead of $[L]_\beta^\beta$.

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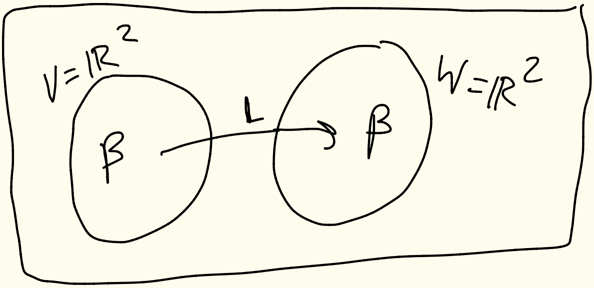
Ex: $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

$V = W = \mathbb{R}^2$
$F = \mathbb{R}$

you can check that L is a linear transformation



Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$.

Let's compute $[L]_\beta = [L]_\beta^\beta$

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

plug β into L

write the result in terms of β

$$\text{So, } [L]_\beta = \left(\left[L\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_\beta \middle| \left[L\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_\beta \right) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

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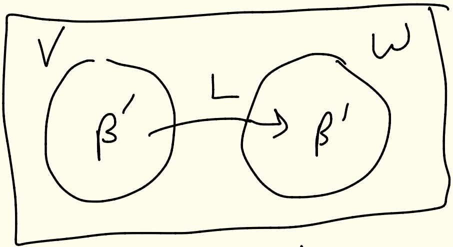
Let $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$ ← you can show this is a basis for $\mathbb{R}^2$

Let's find $[L]_{\beta'}$.



Recall:

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$$

$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\underbrace{L\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\text{plug } \beta' \text{ into } L} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \underbrace{b\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\text{write answer in terms of } \beta'}$$

Need to solve

$$\boxed{\begin{aligned} 2 &= a - c \\ 1 &= a + c \\ 0 &= b - d \\ -3 &= b + d \end{aligned}} \Rightarrow \boxed{\begin{aligned} a &= \tfrac{3}{2} \\ c &= -1/2 \\ b &= -3/2 \\ d &= -3/2 \end{aligned}}$$

So,

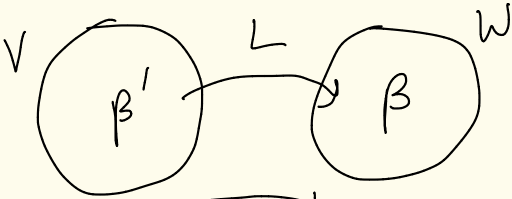
$$[L]_{\beta'} = \left(\left[L\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{\beta'} \middle| \left[L\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]_{\beta'} \right)$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$$

Let's now calculate $[L]_{\beta'}^{\beta}$

pg 9



$$\beta' = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right]$$

$$\beta = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$\boxed{L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\2x-y\end{pmatrix}}$$

$$\underbrace{L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}}\ = \ 2\cdot\begin{pmatrix}1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$\underbrace{L\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\-3\end{pmatrix}}_{\text{plug } \beta' \text{ into } L} = \underbrace{0\cdot\begin{pmatrix}1\\0\end{pmatrix} - 3\cdot\begin{pmatrix}0\\1\end{pmatrix}}_{\text{write the answers in terms of } \beta}$$

So,

$$[L]_{\beta'}^{\beta} = \left(\left[L\begin{pmatrix}1\\1\end{pmatrix}\right]_{\beta} \middle| \left[L\begin{pmatrix}-1\\1\end{pmatrix}\right]_{\beta} \right)$$

$$= \begin{pmatrix}2 & 0\\1 & -3\end{pmatrix}$$

Math 4570

10/12/20

Ex: (Continued from last time)

Recap: $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

$\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$, $[L]_\beta = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

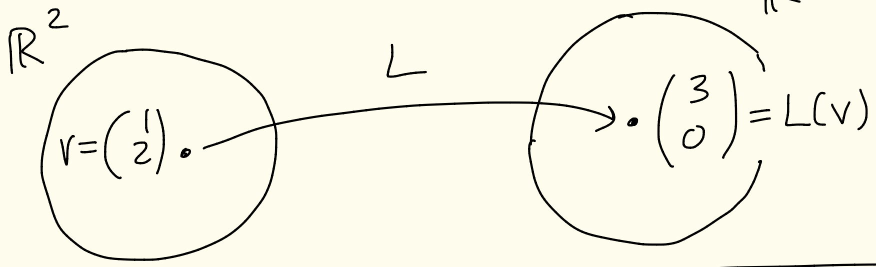
$\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]$, $[L]_{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$

$$[L]_{\beta'}^{\beta} = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$$

What do these matrices do?

Recall: $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



We are using the standard basis here $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$.

$$[L]_\beta [v]_\beta = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2-2 \end{pmatrix} = \overbrace{\underbrace{\begin{pmatrix} 3 \\ 0 \end{pmatrix}}_{\text{coordinates of } L(v) \text{ w/ respect to } \beta}}^{[L(v)]_\beta}$$

$$[v]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{[L]_\beta} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [L]_\beta [v]_\beta$$

$$v = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 0 \end{pmatrix}}_{L(v)}$$

So: $[L]_\beta [v]_\beta = [L(v)]_\beta$

pg 3

Let's now use $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$.

$$[L]_{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$$

this matrix will compute L, but it takes β' coordinates as input and it outputs β' cousdinates

That is,

$$[L]_{\beta'} [v]_{\beta'} = [L(v)]_{\beta'}$$

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Let's find $[v]_{\beta'}$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = v = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So need to solve

$$\boxed{\begin{aligned} 1 &= \alpha_1 - \alpha_2 \\ 2 &= \alpha_1 + \alpha_2 \end{aligned}} \longleftarrow \boxed{\begin{aligned} \alpha_1 &= \frac{3}{2} \\ \alpha_2 &= 1/2 \end{aligned}}$$

Thus, $v = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

So, $[v]_{\beta'} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

So,

$$[L]_{\beta'} [v]_{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} =$$

$$= \begin{pmatrix} 9/4 - 3/4 \\ -3/4 - 3/4 \end{pmatrix} = \begin{pmatrix} 6/4 \\ -6/4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

$\boxed{\text{pg } 4}$

Supposedly then $[L(v)]_{\beta'} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$.

Let's verify.

$$\frac{3}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 + 3/2 \\ 3/2 - 3/2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$= L(v)$$

So, $[L]_{\beta'} [v]_{\beta'} = [L(v)]_{\beta'}$

What does $[L]^{\beta}_{\beta'}$ do?

It turns out that

$$[L]^{\beta}_{\beta'}[v]_{\beta'} = [L(v)]_{\beta}$$

So, $[L]^{\beta}_{\beta'}$ computes L but it takes as input β'-coordinates and it outputs β-coordinates.

For example

$$[v]_{\beta'} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} \quad \text{and} \quad [L]^{\beta}_{\beta'} = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$$

$$[L]^{\beta}_{\beta'}[v]_{\beta'} = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

β-coordinates

$$= \begin{pmatrix} 3+0 \\ \frac{3}{2} - \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [L(v)]_{\beta}$$

$$\boxed{L(v) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow [L(v)]_{\beta} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}}$$

<u>Theorem:</u> Let V and W be finite-dimensional vector spaces over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V and $\beta' = [w_1, w_2, \ldots, w_m]$ be an ordered basis for W. Let $L: V \to W$ be a linear transformation.

Then,
$$[L]_\beta^{\beta'} [x]_\beta = [L(x)]_{\beta'} \quad \text{for all } x \in V.$$

<u>proof:</u> Since β is a basis we may write $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, where $\alpha_i \in F$.

So,
$$[x]_\beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Since β' is a basis we may write

$$
\begin{aligned}
L(v_1) &= a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m \\
L(v_2) &= a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m \\
&\vdots \\
L(v_n) &= a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m
\end{aligned}
$$

where $a_{ij} \in F$.

So,

$$
[L]_{\beta}^{\beta'} = \left([L(v_1)]_{\beta'} \,\middle|\, [L(v_2)]_{\beta'} \,\middle|\, \cdots \,\middle|\, [L(v_n)]_{\beta'} \right)
$$

$$
= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}
$$

We have that

$$
\begin{aligned}
L(x) &= L(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) \\
&= \alpha_1 L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_n L(v_n)
\end{aligned}
$$

L is linear

$$
\begin{aligned}
&= \alpha_1 \big(\overbrace{a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m}^{L(v_1)}\big) \\
&\quad + \alpha_2 \big(\underbrace{a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m}_{L(v_2)}\big) \\
&\quad + \dots + \\
&\quad + \alpha_n \big(\underbrace{a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m}_{L(v_n)}\big)
\end{aligned}
$$

$$
\begin{aligned}
&= (\alpha_1 a_{11} + \alpha_2 a_{12} + \dots + \alpha_n a_{1n}) w_1 \\
&\quad + (\alpha_1 a_{21} + \alpha_2 a_{22} + \dots + \alpha_n a_{2n}) w_2 \\
&\quad + \dots + \\
&\quad + (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \dots + \alpha_n a_{mn}) w_m
\end{aligned}
$$

So,

$$[L(x)]_{\beta'} = \begin{pmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$= [L]_{\beta}^{\beta'} [x]_{\beta}$$

$\blacksquare$

Now we show how to make a matrix that changes one coordinate system into another.

<u>Def:</u> Let V be a finite-dimensional vector space over a field F. Let β and β' be ordered bases for V. Let $I: V \to V$ be the <u>identity linear transformation</u> where $I(x) = x$ for all $x \in V$. The matrix $[I]_{\beta}^{\beta'}$ is called the <u>change of basis matrix from β to β'</u>.

<u>Ex:</u> Let $V=\mathbb{R}^2$, $F=\mathbb{R}$.

Let $\beta = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$

and $\beta' = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right]$ as earlier.

Let's calculate $[I]_{\beta}^{\beta'}$

you could solve a system to get these #s

$$I\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\underbrace{I\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}}_{\text{plug } \beta \text{ into } I} = \underbrace{\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}}_{\text{write the answer in terms of } \beta'}$$

$$[I]_{\beta}^{\beta'} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Math 4570

10/14/20

Test 1 is on Weds October 21st,

See 10/7 notes for calculations to know

There won't be class on the test day

Instructions /info are on the next page →

Structure

1. calculation type questions /subgroup
2. proofs

Math 4570 - Fall 2020 - Test 1

- You will receive the exam on Wednesday, October 21st at 8am.
- I will email it to you and also post the test on the class website.
  website: https://www.calstatela.edu/research/ashahee/
  click on Math 4570
- Please send your solutions back to me by Thursday, October 22nd at 12pm in the afternoon.
- Pick a consecutive 2-hour window to take this exam, such as 2pm - 4pm. You may only use 2 hours of consecutive time. Do not split the time (like 12-1pm and then 5-6pm).
- You can only use your mind to take this exam. No help from any sources or people. No books, no notes, no online, etc.
- No calculators.
- Use blank paper (like printer paper) if you have it please.
- On the first page of your exam, before any of your solutions, put these three things:
  (a) Your name.
  (b) The time period that you chose (such as 2pm - 4pm on Wednesday)
  (c) Copy this statement and then sign your signature after it:

  "Everything on this test is my own work. I did not use any sources or talk to anyone about this exam." **your signature** __________
- After your name and the above statement with signature, start putting your solutions to the problems. Please put them in order. That is, first problem 1, then problem 2, etc. You can put each one on its own page.
- Please scan your test using a scanner (such as a free one on your phone) and put it into one pdf document with your problems in order.
- To get a clean scan, make sure there is plenty of light, the phone is held flat directly above the paper, and the paper is placed on a flat object such as the floor or a table.
- When your 2 hour time period is done, please scan your test and email me a pdf to ashahee@calstatela.edu.

The problems are on the next page.

Ex: (Continued from last time)

$V=\mathbb{R}^2$, $F=\mathbb{R}$

$\beta=\left[\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right]$

$\beta'=\left[\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\right]$

$[I]_\beta^{\beta'}=\begin{pmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{pmatrix}$

from last time

What $[I]_\beta^{\beta'}$ does is it turns β-coordinates into β'-coordinates.

For example, in a previous example we looked at $v=\begin{pmatrix}1\\2\end{pmatrix}$ and saw that

$v=\begin{pmatrix}1\\2\end{pmatrix}=1\cdot\begin{pmatrix}1\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\1\end{pmatrix}$ so $[v]_\beta=\begin{pmatrix}1\\2\end{pmatrix}$

$v=\begin{pmatrix}1\\2\end{pmatrix}=\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}$ so $[v]_{\beta'}=\begin{pmatrix}3/2\\1/2\end{pmatrix}$

Note that

$$\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+1 \\ -\frac{1}{2}+1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

So, $[I]_{\beta}^{\beta'}[v]_{\beta} = [v]_{\beta'}$ for this particular v

---

<u>Thm:</u> Let V be a finite-dimensional vector space over a field F and let β and β' be ordered bases for V. Let $[I]_{\beta}^{\beta'}$ be the change of basis matrix from β to β'. Then $[I]_{\beta}^{\beta'}[x]_{\beta} = [x]_{\beta'}$ for all $x \in V$.

<u>proof:</u> From thm from last class

$$[I]_{\beta}^{\beta'}[x]_{\beta} = [I(x)]_{\beta'} = [x]_{\beta'} \quad \blacksquare$$

$[L]_{\beta}^{\beta'}[x]_{\beta} = [L(x)]_{\beta'}$

$I(x) = x$
$\forall x \in V$

<u>Def:</u> Let V be a finite-dimensional vector space over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V. Define $\Phi : V \to F^n$ by $\Phi(x) = [x]_\beta$.

[Note that Φ depends on β, so sometimes we write Φ_β for Φ, or sometimes not when β is understood or given.]

We call Φ a <u>canonical isomorphism between V and F^n</u>.

pg 5

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$, $\beta = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$

$$\Phi\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_\beta = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

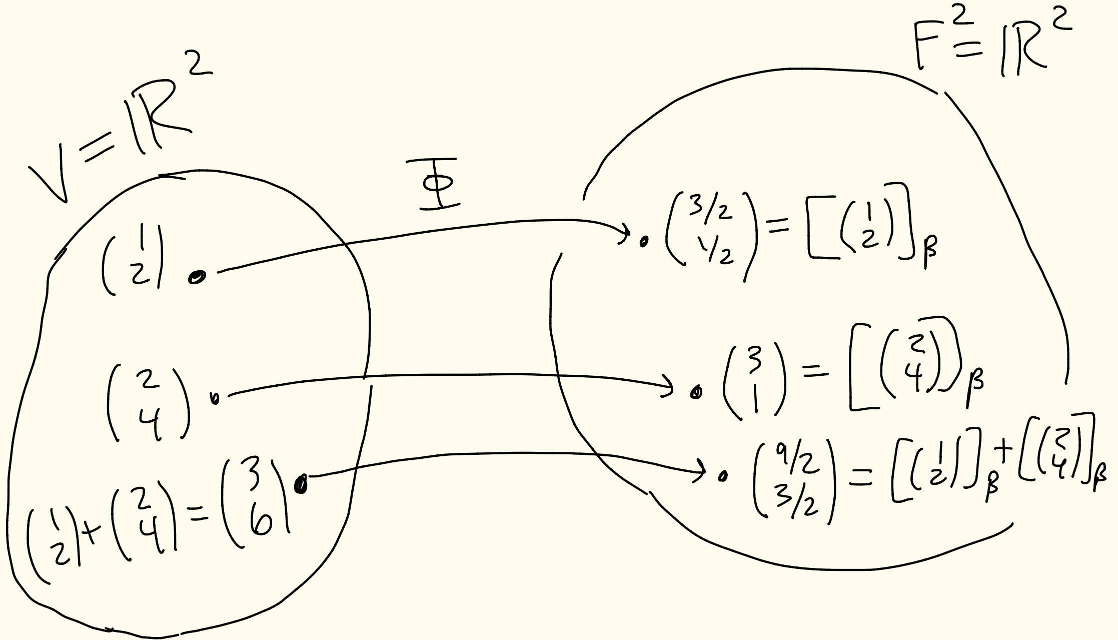
$$\boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$\Phi\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \left[\begin{pmatrix} 2 \\ 4 \end{pmatrix}\right]_\beta = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$\Phi\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \left[\begin{pmatrix} 3 \\ 6 \end{pmatrix}\right]_\beta = \begin{pmatrix} 9/2 \\ 3/2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \frac{9}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$



Φ is an isomorphism

This follows from the thm we proved about how linear transformations are constructed.

$\Phi : V \to F^n$

$\beta = [v_1, v_2, \ldots, v_n]$ is an ordered basis for V

Pick $\beta' = \left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right]$ to be the standard basis for F^n.

Then,

$$\Phi(v_1) = \Phi(1 \cdot v_1 + 0v_2 + \cdots + 0v_n) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Phi(v_2) = \Phi(0v_1 + 1v_2 + \cdots + 0v_n) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi(v_n) = \Phi(0v_1 + 0v_2 + \cdots + 1v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Also, if $x = \alpha_1 v_1 + \dots + \alpha_n v_n \in V$

then

$$\Phi(x) = \Phi(\alpha_1 v_1 + \dots + \alpha_n v_n)$$
$$= \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$
$$= \alpha_1 \Phi(v_1) + \alpha_2 \Phi(v_2) + \dots + \alpha_n \Phi(v_n)$$

We had a thm that says this shows Φ is a linear transformation and since Φ maps β onto a basis β', this implies Φ is an isomorphism.

So, $\boxed{V \cong F^n}$ given by Φ.

Commutative diagram

$L: V \to W$ is a linear transformation

V & W are finite dimensional vector spaces over F

β and γ are ordered bases for V and W respectively

$\dim(V) = n$ and $\dim(W) = m$

$$
\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & L(x) \arrow[ddd, mapsto] \\
& V \arrow[r, "L"] \arrow[d, "\Phi_\beta"', "\cong"] & W \arrow[d, "\Phi_\gamma", "\cong"'] & \\
& F^n \arrow[r, "{[L]_\beta^\gamma}"] & F^m & \\
{[x]_\beta} \arrow[rrr, mapsto] & & & {[L]_\beta^\gamma [x]_\beta = [L(x)]_\gamma}
\end{tikzcd}
$$

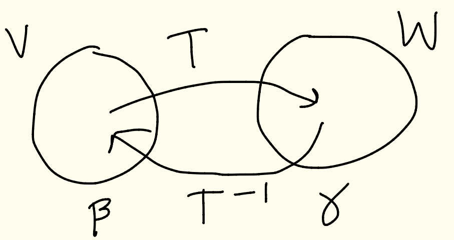
Theorem: Let V and W be finite dimensional vector spaces over a field F, Let $T: V \to W$ be a linear transformation, Let β and γ be ordered bases for V and W, respectively.

T is an isomorphism iff

$[T]_\beta^\gamma$ is invertible.

Furthermore, if this is so then

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}.$$



proof:

($\Longrightarrow$) Suppose that T is an isomorphism.
So, T is 1-1 and onto and
$\dim(V) = \dim(W)$.

Let $n = \dim(V) = \dim(W)$.

So, $[T]_\beta^\gamma$ is an $n \times n$ matrix.

Let I_n be the $n \times n$ identity matrix
and let $I_V : V \to V$ and
$I_W : W \to W$ be the identity
linear transformations, where
$I_V(x) = x$ for all $x \in V$
and $I_W(x) = x$ for all $x \in W$.

Then,

$$[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = [T^{-1} \circ T]_\beta^\beta = [I_V]_\beta^\beta = I_n$$

HW 4 3(a) $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

HW

and

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [T \circ T^{-1}]_\gamma^\gamma = [I_W]_\gamma^\gamma = I_n.$$

Thus,

$[T]_\beta^\gamma$ is an invertible matrix

and $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$

$$\boxed{(\Longrightarrow)}$$

Math 4570

10/19/20

⑦ Let V be a vector space over a field F. Let W_1 and W_2 be subspaces of V.

Define
$$W_1 + W_2 = \{ x + y \mid x \in W_1, y \in W_2 \}$$

(a) Show $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

pf of (a):

Let $x \in W_1$.
Since W_2 is a subspace, $\vec{0} \in W_2$.
Then $x = \underbrace{x}_{\text{in } W_1} + \underbrace{\vec{0}}_{\text{in } W_2} \in W_1 + W_2$

So, $W_1 \subseteq W_1 + W_2$.
Also let $y \in W_2$. Since W_1 is a subspace, $\vec{0} \in W_1$. Thus,
$y = \vec{0} + y \in W_1 + W_2$. So, $W_2 \subseteq W_1 + W_2$.

(b) $W_1 + W_2$ is a subspace of V.

<u>proof:</u>

- Since W_1 and W_2 are subspaces, $\vec{0} \in W_1$ and $\vec{0} \in W_2$.
  Thus, $\vec{0} = \underbrace{\vec{0}}_{\text{in } W_1} + \underbrace{\vec{0}}_{\text{in } W_2} \in W_1 + W_2$

- Let $a, b \in W_1 + W_2$ and $\alpha \in F$.
  Then $a = x_1 + y_1$ and $b = x_2 + y_2$
  where $x_1, x_2 \in W_1$ and $y_1 + y_2 \in W_2$
  Since $x_1, x_2 \in W_1$ and W_1 is a subspace
  we have that $x_1 + x_2 \in W_1$.
  Since $y_1, y_2 \in W_2$ and W_2 is a subspace
  we have that $y_1 + y_2 \in W_2$.

  So,
  $$a + b = x_1 + y_1 + x_2 + y_2$$
  $$= \underbrace{(x_1 + x_2)}_{\text{in } W_1} + \underbrace{(y_1 + y_2)}_{\text{in } W_2} \in W_1 + W_2$$

pg 3

Since $x_1 \in W_1$ and $y_1 \in W_2$, and W_1 & W_2 are subspaces we have $\alpha x_1 \in W_1$ and $\alpha y_1 \in W_2$.

So,
$$\alpha a = \alpha(x_1 + y_1) = \underbrace{\alpha x_1}_{\text{in } W_1} + \underbrace{\alpha y_1}_{\text{in } W_2} \in W_1 + W_2$$

By the above $W_1 + W_2$ is a subspace of V. $\blacksquare$

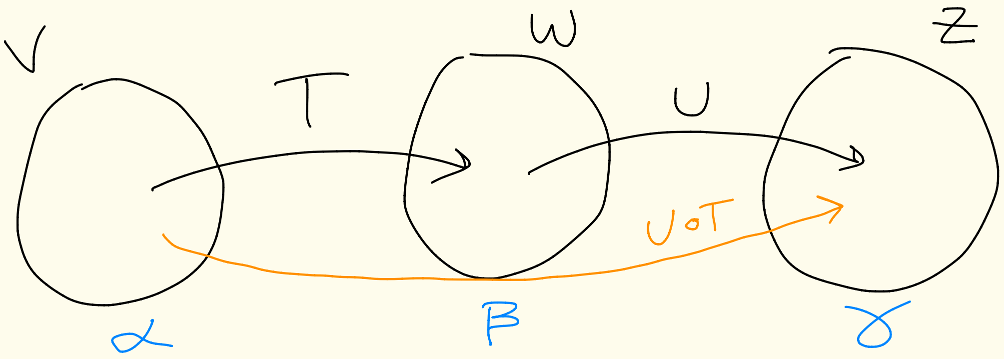
HW 4 #3

Let V, W, and Z be finite dimensional vector spaces over a field F. Let α, β, and γ be ordered bases for V, W, and Z respectively.

Let $T: V \to W$ and $U: W \to Z$ be linear transformations.

(a) $U \circ T : V \to Z$ is a linear transformation

(b) $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$



proof of (a)

Let $v_1, v_2 \in V$ and $c_1, c_2 \in F$.

Then,

$$(U \circ T)(c_1 v_1 + c_2 v_2)$$

$$= U(T(c_1 v_1 + c_2 v_2))$$

$$= U(c_1 T(v_1) + c_2 T(v_2))$$

T is linear

$$= c_1 U(T(v_1)) + c_2 U(T(v_2))$$

U is linear

$$= c_1 (U \circ T)(v_1) + c_2 (U \circ T)(v_2)$$

So, $U \circ T$ is linear.

proof of (b): $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

Let $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n]$,

$\beta = [\beta_1, \beta_2, \dots, \beta_m]$,

$\gamma = [\gamma_1, \gamma_2, \dots, \gamma_p]$.

Let's calculate $[T]_\alpha^\beta$.

Suppose

$$\begin{aligned} T(\alpha_1) &= t_{11}\beta_1 + t_{21}\beta_2 + \dots + t_{m1}\beta_m \\ T(\alpha_2) &= t_{12}\beta_1 + t_{22}\beta_2 + \dots + t_{m2}\beta_m \\ &\vdots \\ T(\alpha_n) &= t_{1n}\beta_1 + t_{2n}\beta_2 + \dots + t_{mn}\beta_m \end{aligned}$$

where $t_{ij} \in F$.

So, $[T]_\alpha^\beta = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix}$

More compact notation: $T(\alpha_i) = \sum_{j=1}^{m} t_{ji}\beta_j$

pg 7

Similarly we have

$$\begin{aligned} U(\beta_1) &= u_{11}\gamma_1 + u_{21}\gamma_2 + \cdots + u_{p1}\gamma_p \\ U(\beta_2) &= u_{12}\gamma_1 + u_{22}\gamma_2 + \cdots + u_{p2}\gamma_p \\ &\vdots \\ U(\beta_m) &= u_{1m}\gamma_1 + u_{2m}\gamma_2 + \cdots + u_{pm}\gamma_p \end{aligned}$$

where $u_{ij} \in F$.

So,

$$[U]_\beta^\gamma = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & & \vdots \\ u_{p1} & u_{p2} & \cdots & u_{pm} \end{pmatrix}$$

Compact form : $U(\beta_j) = \sum_{k=1}^{p} u_{kj}\gamma_k$

Then,

$$(U \circ T)(\alpha_i)$$

$$= U(T(\alpha_i))$$

$$= U\left(\sum_{j=1}^{m} t_{ji} \beta_j\right)$$

$$\underset{\boxed{U \text{ is linear}}}{=} \sum_{j=1}^{m} t_{ji}\, U(\beta_j) = \sum_{j=1}^{m} t_{ji} \sum_{k=1}^{p} u_{kj} \gamma_k$$

$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{m} t_{ji} u_{kj} \right) \gamma_k$$

$$= \underbrace{\sum_{k=1}^{p} \left(\sum_{j=1}^{m} u_{kj} t_{ji} \right) \gamma_k}_{\text{give us the ith column of } [U \circ T]_\alpha^\gamma}$$

The element in the k-th row and ith column of $[U \circ T]_\alpha^\gamma$ is $\sum_{j=1}^{m} u_{kj} t_{ji}$

And,

$$[U]_\beta^\gamma [T]_\alpha^\beta = \underbrace{\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & & & \vdots \\ u_{p1} & u_{p2} & \cdots & u_{pm} \end{pmatrix}}_{p \times m} \underbrace{\begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & & & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix}}_{m \times n}$$

$$= \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{pmatrix}}_{p \times n}$$

dot product

Where $c_{ki} = \begin{pmatrix} k\text{th row from} \\ [U]_\beta^\gamma \end{pmatrix} \cdot \begin{pmatrix} i\text{th column} \\ \text{from} \\ [T]_\alpha^\beta \end{pmatrix}$

$$= u_{k1} t_{1i} + u_{k2} t_{2i} + \cdots + u_{km} t_{mi}$$

$$= \sum_{j=1}^{m} u_{kj} t_{ji} .$$

Thus, $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$ $\blacksquare$

Let V and W be vector spaces over a field F. Let α and β be ordered bases for V and W, respectively. Let $T_1 : V \to W$ and $T_2 : V \to W$. If $[T_1]_\alpha^\beta = [T_2]_\alpha^\beta$, then $T_1 = T_2$.

proof: Let $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ and $\beta = [\beta_1, \beta_2, \ldots, \beta_m]$.

Suppose

$$[T_1]_\alpha^\beta = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & & & \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix} = [T_2]_\alpha^\beta.$$

Then,

$$T_1(\alpha_1) = t_{11}\beta_1 + t_{21}\beta_2 + \ldots + t_{m1}\beta_m = T_2(\alpha_1)$$

$$T_1(\alpha_2) = t_{12}\beta_1 + t_{22}\beta_2 + \ldots + t_{m2}\beta_m = T_2(\alpha_2)$$

$$\vdots$$

$$T_1(\alpha_n) = t_{1n}\beta_1 + t_{2n}\beta_2 + \ldots + t_{mn}\beta_m = T_2(\alpha_n)$$

So, $T_1(\alpha_i) = T_2(\alpha_i)$ for $i = 1, 2, \ldots, n$. pg 11

Let $x \in V$.

Write $x = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n$
for some $c_i \in F$.

Then,

$$T_1(x) = T_1(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n)$$

T_1 linear

$$= c_1T_1(\alpha_1) + c_2T_1(\alpha_2) + \ldots + c_nT_1(\alpha_n)$$

$T_1(\alpha_i) = T_2(\alpha_i)$

$$= c_1T_2(\alpha_1) + c_2T_2(\alpha_2) + \ldots + c_nT_2(\alpha_n)$$

T_2 is linear

$$= T_2(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n)$$

$$= T_2(x).$$

So, $T_1 = T_2$. $\square$

Math 4570

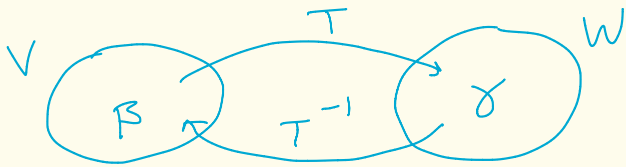
10/26/20

Goal: Finish HW 4 material today.

Theorem: Let V and W be finite-dimensional vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Let β and γ be ordered bases for V and W, respectively.

T is an isomorphism/invertible iff $[T]_\beta^\gamma$ is invertible.

Furthermore, if this is the case then $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$



We stated and proved one direction two weeks ago. Look in notes on website if you want to see the proof

Corollary: Let V be a finite-dimensional vector space over a field F. Let β and β' be ordered basis for V. Let $Q = [I]_{\beta}^{\beta'}$ be the change of basis matrix from β to β'. Here $I : V \to V$ where $I(x) = x$ for all $x \in V$.

Then:

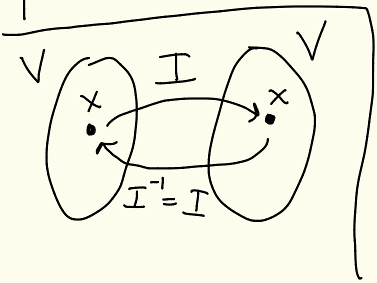
① Q is invertible and $Q^{-1} = [I]_{\beta'}^{\beta}$

② If $T : V \to V$ is a linear transformation, then $[T]_{\beta} = \underbrace{Q^{-1}[T]_{\beta'}Q}_{[I]_{\beta'}^{\beta}[T]_{\beta'}[I]_{\beta}^{\beta'}}$

---

proof of ①: I is invertible and $I^{-1} = I$. Thus, $Q = [I]_{\beta}^{\beta'}$ is invertible and

$$Q^{-1} = [I^{-1}]_{\beta'}^{\beta} = [I]_{\beta'}^{\beta}.$$



Proof of ②:

We have that

$$Q^{-1}[T]_{\beta'}Q = [I]_{\beta'}^{\beta}\overbrace{[T]_{\beta'}[I]_{\beta}^{\beta'}}^{[T]_{\beta'}^{\beta'}}$$

$$= [I]_{\beta'}^{\beta}[\underbrace{T\circ I}_{\boxed{T\circ I = T}}]_{\beta}^{\beta'}$$

HW: $\boxed{[U\circ T]_{\alpha}^{\gamma} = [U]_{\delta}^{\gamma}[T]_{\alpha}^{\delta}}$

$$= [I]_{\beta'}^{\beta}[T]_{\beta}^{\beta'}$$

$$= [\underbrace{I\circ T}_{\boxed{I\circ T = T}}]_{\beta}^{\beta}$$

$$= [T]_{\beta}^{\beta} = [T]_{\beta}.$$

$\blacksquare$

<u>Def:</u> Let A and B be $n \times n$ matrices with entries from a field F. We say that A and B are <u>similar</u> if there exists an $n \times n$ invertible matrix Q with entries from F where

$$B = Q^{-1} A Q.$$

In the previous thm we saw that $[T]_\beta$ and $[T]_{\beta'}$ are similar matrices

---

Theorem: Let V be a finite-dimensional vector space over a field F. Let β be an ordered basis for V. Let $T: V \to V$ be a linear transformation. Suppose $\dim(V) = n$. If A is an $n \times n$ matrix with entries from F that is similar to $[T]_\beta$, then $A = [T]_\gamma$ where γ is some ordered basis for V.

---

proof: Let $n = \dim(V)$. Then $[T]_\beta$ is $n \times n$. Let $\beta = [v_1, v_2, \ldots, v_n]$.
Since A is similar to $[T]_\beta$ there exists an invertible $n \times n$ matrix Q with entries from F where $A = Q^{-1}[T]_\beta Q$.

Let Q_{ij} denote the ij-th entry of the matrix Q.

That is, $Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & & & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix}$.

Define the vectors $w_1, w_2, \ldots, w_n$ by the equations

$$w_j = \underbrace{\sum_{i=1}^{n} Q_{ij} V_j}_{\text{this sum runs over the jth column of } Q}$$

Let $\gamma = [w_1, w_2, \ldots, w_n]$.

We will now show γ is a basis for V. We do this by showing that γ is a lin. ind. set. Then since $\dim(V) = n$, and γ has n elements, it must be a basis for V.

Suppose

$$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_n w_n = \vec{0}$$

where $\alpha_i \in F$. Then,

$$\begin{aligned} &\alpha_1 (Q_{11} v_1 + Q_{21} v_2 + \cdots + Q_{n1} v_n) \\ &+ \alpha_2 (Q_{12} v_1 + Q_{22} v_2 + \cdots + Q_{n2} v_n) \\ &+ \cdots \\ &+ \alpha_n (Q_{1n} v_1 + Q_{2n} v_2 + \cdots + Q_{nn} v_n) = \vec{0} \end{aligned}$$

Rearranging we get

$$\begin{aligned} &(\alpha_1 Q_{11} + \alpha_2 Q_{12} + \cdots + \alpha_n Q_{1n}) v_1 \\ &+ (\alpha_1 Q_{21} + \alpha_2 Q_{22} + \cdots + \alpha_n Q_{2n}) v_2 \\ &+ \cdots + (\alpha_1 Q_{n1} + \alpha_2 Q_{n2} + \cdots + \alpha_n Q_{nn}) v_n = \vec{0} \end{aligned}$$

Since $\beta = [v_1, v_2, \ldots, v_n]$ is a lin. ind. set we get

$$\begin{aligned} \alpha_1 Q_{11} + \alpha_2 Q_{12} + \cdots + \alpha_n Q_{1n} &= 0 \\ \alpha_1 Q_{21} + \alpha_2 Q_{22} + \cdots + \alpha_n Q_{2n} &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ \alpha_1 Q_{n1} + \alpha_2 Q_{n2} + \cdots + \alpha_n Q_{nn} &= 0 \end{aligned}$$

Thus,

$$\begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & & & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or

$$Q \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since Q^{-1} exists we get

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \underbrace{Q^{-1}Q}_{I} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus, $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$.

Thus, γ is a lin. ind. set.

So, γ is a basis.

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By the definition of w_j, Q is the change of basis matrix $[I]_\gamma^\beta$.

$$w_j = \sum_{i=1}^{n} Q_{ij} v_i$$

$$I(w_j) = I\left(\sum_{i=1}^{n} Q_{ij} v_i\right) = \sum_{i=1}^{n} Q_{ij} v_i$$

So, the jth column of $[I]_\gamma^\beta$ is $\begin{pmatrix} Q_{1j} \\ Q_{2j} \\ \vdots \\ Q_{nj} \end{pmatrix}$ = jth column of Q

Hence, $Q^{-1} = [I]_\beta^\gamma$. And,

$$A = Q^{-1}[T]_\beta Q = [I]_\beta^\gamma [T]_\beta [I]_\gamma^\beta = [T]_\gamma \quad \blacksquare$$

Math 4570

10/28/20

Review of determinants

(In between HW 4/HW 5)

We will define determinants recursively.

<u>Def:</u> Let A be an $n \times n$ matrix with coefficients from a field F. Let $1 \leq i, j \leq n$. The matrix A_{ij} is defined to be the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and jth column of A

<u>Ex:</u>

$$A = \begin{pmatrix} \pi & -\frac{1}{10} & 1 \\ 0 & 5 & -2 \\ i & \sqrt{2} & 3 \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} 5 & -2 \\ \sqrt{2} & 3 \end{pmatrix}$$

$$\begin{pmatrix} \pi & -\frac{1}{10} & 1 \\ 0 & 5 & -2 \\ i & \sqrt{2} & 3 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} \pi & -\frac{1}{10} \\ i & \sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \pi & -\frac{1}{10} & 1 \\ 0 & 5 & -2 \\ i & \sqrt{2} & 3 \end{pmatrix}$$

<u>Def:</u> Let A be an $n \times n$ matrix with entries from a field F. Let a_{ij} be the entry in the ith row and jth column of A.

① If $n=1$ and $A = (a_{11})$, then define $\det(A) = a_{11}$

② If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then

define $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

③ If $n \geq 3$, then define $\det(A)$ as follows. Pick a column j $(1 \leq j \leq n)$

Define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\substack{\text{sum over rows} \\ \text{column } j \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(\underbrace{A_{ij}}_{\substack{\text{this is} \\ \text{an} \\ (n-1)\times(n-1) \\ \text{matrix}}})$$

This is called the <u>expansion of the determinant along the jth column</u>

Note: One can also expand along a row in part (3). You pick a row i $(1 \le i \le n)$ and replace step (3) with

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\substack{\text{sum over} \\ \text{the columns} \\ \text{row } i \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Fact: This def is well-defined.

One can show that the final result is the same no matter what row or column you expand on in each step

---

Notation: One also uses bars in the notation. For example,

$$\det\begin{pmatrix} 10 & 1 \\ -1 & 5 \end{pmatrix} = \begin{vmatrix} 10 & 1 \\ -1 & 5 \end{vmatrix}$$

Ex: $\det(-3) = -3$

Ex: $\det\begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix} = (1)(3) - (5)(-1) = 8$

Ex: $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

Expand on row $i=1$: $\begin{pmatrix} \boxed{3 \quad 1 \quad 0} \\ -2 \quad -4 \quad 3 \\ 5 \quad 4 \quad -2 \end{pmatrix}$

$$\det(A) = (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+3} a_{13} \det(A_{13})$$

$$= \underbrace{(1)(3) \cdot \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix}}_{\begin{vmatrix} \overset{a_{11}}{\cancel{3}} & \cancel{1} & \cancel{0} \\ \cancel{-2} & -4 & 3 \\ \cancel{5} & 4 & -2 \end{vmatrix}} + \underbrace{(-1)(1) \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix}}_{\begin{vmatrix} \cancel{3} & \overset{a_{12}}{\cancel{1}} & \cancel{0} \\ -2 & \cancel{-4} & 3 \\ 5 & \cancel{4} & -2 \end{vmatrix}} + \underbrace{(1)(0) \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}}_{\begin{vmatrix} \cancel{3} & \cancel{1} & \overset{a_{13}}{\cancel{0}} \\ -2 & -4 & \cancel{3} \\ 5 & 4 & \cancel{-2} \end{vmatrix}}$$

$$= 3[8-12] - [4-15] + 0 = -1$$

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$$\left((-1)^{i+j}\right) = \begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

<u>Useful tool:</u>

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

always put a + in top left and alternate +/−

Do the same determinant but expand on column 2

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \quad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (-1)(1)\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + (1)(-4)\begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix} + (-1)(4)\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} \quad \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} \quad \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$$

$$= -[4-15] - 4[-6-0] - 4[9-0] = -1$$

For 4x4, the +/- matrix would be

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$$= (1)(1)\underbrace{\begin{vmatrix} 3 & -1 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}}_{\begin{vmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{vmatrix}} + (-1)(0)\underbrace{\begin{vmatrix} 2 & 0 & -1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix}}_{\begin{vmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{vmatrix}}$$

$$+ (1)(0)\underbrace{\begin{vmatrix} 2 & 0 & -1 \\ 3 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix}}_{\begin{vmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{vmatrix}} + (-1)(0)\underbrace{\begin{vmatrix} 2 & 0 & -1 \\ 3 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix}}_{\begin{vmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{vmatrix}} =$$

$$= \begin{vmatrix} 3 & -1 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 0$$

$$= 3[1+2] + 1[2+3]$$

$$= 9 + 5 = 14$$

<u>Properties of the determinant:</u>

Let F be a field and A and B be $n \times n$ matrices with entries from F. Then:

① $\det(AB) = \det(A)\det(B)$

② A is invertible iff $\det(A) \neq 0$.

If A is invertible, then

$$\det(A^{-1}) = (\det(A))^{-1}$$

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Test 1 solutions on canvas.

You're grade will the max of these two systems:

syllabus system: $\frac{1}{3}$ test 1, $\frac{1}{3}$ test 2, $\frac{1}{3}$ final

other system: $\frac{1}{2}$ - $\max\{$test 1, test 2$\}$

$\frac{1}{2}$ - final

Test 2 is on

Weds 11/18.

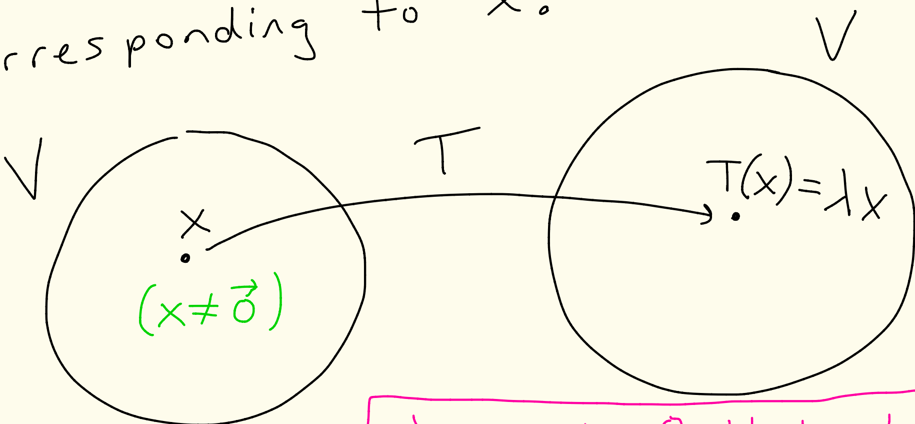
Test 2 covers

HW 3 and HW 4.

Eigenvalues, Eigenvectors, and Diagonalization

(HW 5)

Def: Let V be a vector space over a field F. Let $T: V \to V$ be a linear transformation. If $x \in V$ with $x \neq \vec{0}$ and $T(x) = \lambda x$ for some $\lambda \in F$, then we call x an eigenvector of T and λ the eigenvalue corresponding to x.



$\boxed{\lambda \text{ can be } 0\text{, thats ok}}$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 4a+2b \end{pmatrix}$$

you can check this is linear

Then,

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 4-2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So, $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = -2$.

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+12 \\ 12+8 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

So, $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = 5$.

Ex: $P_2(\mathbb{R}) = \{a+bx+cx^2 \mid a,b,c \in \mathbb{R}\}$

$$\left.\begin{aligned} &T : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \\ &T(a+bx+cx^2) = b+2cx \end{aligned}\right\} \text{you can check } T \text{ is linear}$$

[That is, $T(f) = f'$]

Then,

$$T(1) = 0 = 0 \cdot 1$$

So, 1 is an eigenvector with eigenvalue 0.

Def: Let V be a finite-dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation. We say that T is diagonalizable if there exists an ordered basis β of V such that $[T]_\beta$ is a diagonal matrix.

Ex: Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 4a+2b \end{pmatrix}$ as on page 3.

We saw that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are eigenvectors of T.

Let $\beta = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]$.

You can check that these two vectors are linearly indenendent and since are two of them and $\dim(\mathbb{R}^2) = 2$, they are a basis for $\mathbb{R}^2$.

Let's calculate $[T]_\beta$.

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\underbrace{T\begin{pmatrix} 3 \\ 4 \end{pmatrix}}_{\text{plug } \beta \text{ into } T} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = \underbrace{0\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}}_{\text{write the answer in terms of } \beta}$$

So, $[T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$. So, T is diagonalizable.

Let's take a closer look at why this is useful.

Let $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Let x be any vector in $\mathbb{R}^2$.

Since $\beta = [v_1, v_2]$ is a basis for $\mathbb{R}^2$

we can write $x = c_1 v_1 + c_2 v_2$

where $c_1, c_2 \in \mathbb{R}$.

Then,

$$
\begin{aligned}
T(x) &= T(c_1 v_1 + c_2 v_2) \\
&= c_1 T(v_1) + c_2 T(v_2) \\
&= c_1(-2v_1) + c_2(5v_2) \\
&= -2c_1 v_1 + 5c_2 v_2
\end{aligned}
$$

In matrix notation,

$$
[T(x)]_\beta = [T]_\beta [x]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} -2c_1 \\ 5c_2 \end{pmatrix}
$$

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. T is diagonalizable iff there exists an ordered basis $\beta = [v_1, v_2, \dots, v_n]$ of V consisting of eigenvectors of T. Moreover, if this is the case then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where λ_i is the eigenvalue corresponding to v_i.

proof: T is diagonalizable

iff there exists an ordered basis of V such $\beta = [v_1, v_2, \dots, v_n]$

that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

for some $\lambda_i \in F$

iff there exists an ordered basis of V such that $\beta = [v_1, v_2, \dots, v_n]$

$$T(v_1) = \lambda_1 v_1 + 0v_2 + \dots + 0v_n$$

$$T(v_2) = 0v_1 + \lambda_2 v_2 + \dots + 0v_n$$

$$\vdots$$

$$T(v_n) = 0v_1 + 0v_2 + \dots + \lambda_n v_n$$

iff there exists an ordered basis $\beta = [v_1, v_2, \dots, v_n]$ of V consisting of eigenvectors with $T(v_i) = \lambda_i v_i$

So λ_i is the eigenvalue for v_i. $\boxed{}$

Why is this useful?

Let $T: V \to V$ be diagonalizable with basis of eigenvectors $\beta = [v_1, v_2, \dots, v_n]$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Let $x \in V$.

Write $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$.

$$
\begin{aligned}
T(x) &= T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) \\
&= c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n) \\
&= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_n \lambda_n v_n
\end{aligned}
$$

How do we find the eigenvalues and eigenvectors?
Let's work on this question.

<u>Theorem:</u> (HW 5 #4)
Let V be a finite-dimensional vector space over a field F.
Let $T: V \to V$ be a linear transformation.
Let β and γ be ordered bases for V.
Then $\det([T]_\beta) = \det([T]_\gamma)$

<u>pf:</u> HW 5 #4.

$\blacksquare$

The previous theorem makes the next definition well-defined.

<u>Def:</u> Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. The <u>determinant</u> of T is defined to be

$$\det(T) = \det([T]_\beta)$$

where β is any ordered basis for V.

Ex: Recall $P_2(\mathbb{R}) = \{a+bx+cx^2 / a,b,c \in \mathbb{R}\}$

Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be defined as $T(f) = f'$

that is $T(a+bx+cx^2) = b + 2cx$.

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Let $\beta = [1, x, x^2]$.

β is an ordered basis for $V = P_2(\mathbb{R})$.

$$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0x^2$$
$$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0x^2$$
$$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0x^2$$

So,

$$\det(T) = \det([T]_\beta) = \det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

expand on 1st column of all zeros

Theorem: Let V be a finite-dimensional vector space over a field F.
Let $T: V \to V$ be a linear transformation
Let $I: V \to V$ be the identity transformation, that is $I(x) = x \ \forall x \in V$.
If β is an ordered basis for V,

then

$$\det(T - \lambda I) = \det([T]_\beta - \lambda I_n)$$

where I_n is the $n \times n$ identity matrix where $\dim(V) = n$.

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Theorem: Let V be a finite dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Then the following are equivalent:

means: if one of ①,②,③ are true then they are all true.

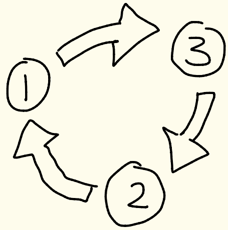
① There exists an eigenvector $x \in V$, $x \neq \vec{0}$, of T with eigenvalue λ

② $\det(T - \lambda I) = 0$

③ $N(T - \lambda I) \neq \{\vec{0}\}$

Here $I: V \to V$ is the identity linear transformation $I(v) = v$ for all $v \in V$.

Proof: We prove



(1 $\Rightarrow$ 3)

Suppose there exists $x \in V$, $x \neq \vec{0}$, with $T(x) = \lambda x$, where $\lambda \in F$.

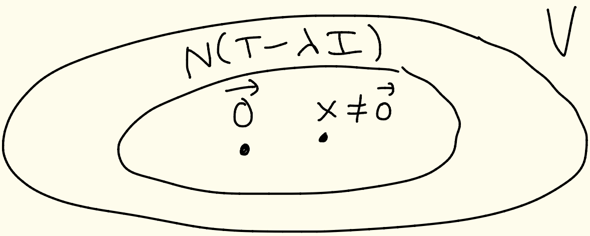
Then, $T(x) = \lambda I(x)$.

So, $T(x) - \lambda I(x) = \vec{0}$.

Hence, $(T - \lambda I)(x) = \vec{0}$

Thus, $x \in N(T - \lambda I)$.

Since $x \neq \vec{0}$, $N(T - \lambda I) \neq \{\vec{0}\}$.



(③ ⇨ ②)

Suppose $N(T-\lambda I) \neq \{\vec{0}\}$.

Then there exists $x \in N(T-\lambda I)$ with $x \neq \vec{0}$.

Note: $\vec{0} \in N(T-\lambda I)$ because $T-\lambda I$ is a linear transformation and HW 3 #1(a) tells us then that $(T-\lambda I)(\vec{0}) = \vec{0}$.

So, $(T-\lambda I)(x) = \vec{0} = (T-\lambda I)(\vec{0})$

Thus, $T-\lambda I$ is not one-to-one.

Hence, $T-\lambda I$ is not invertible.

By HW 5 #5a, $\det(T-\lambda I) = 0$.

(② ⟹ ①)

Suppose $\det(T-\lambda I)=0$.

By Hw 5 #5a, $T-\lambda I$ is not invertible.

So, $T-\lambda I$ is not one-to-one.

By Hw 3 #6a, $N(T-\lambda I)\neq\{\vec{0}\}$

Why? Since $T-\lambda I$ is not 1-1 we have $(T-\lambda I)(x_1)=(T-\lambda I)(x_2)$, where $x_1\neq x_2$.

Then, $(T-\lambda I)(x_1)-(T-\lambda I)(x_2)=\vec{0}$

So, $(T-\lambda I)(x_1-x_2)=\vec{0}$

use linear trans. properties

So, $x_1-x_2\in N(T-\lambda I)$

Since $x_1-x_2\neq\vec{0}$ (because $x_1\neq x_2$), $N(T-\lambda I)\neq\{\vec{0}\}$

So, there exists $x\in V$, $x\neq\vec{0}$, with $x\in N(T-\lambda I)$.

So, $(T-\lambda I)(x)=\vec{0}$.

Thus, $T(x)-\lambda I(x)=\vec{0}$

So, $T(x)=\lambda I(x)$ $\quad I(x)=x$

Ergo, $T(x)=\lambda x$

Thus, $x\neq\vec{0}$ is an eigenvector of T with eigenvalue λ. $\blacksquare$

Theorem: Let V be a finite dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation. Let $I : V \to V$ be the identity transformation, where $I(x) = x$ for all $x \in V$.

Let I_n be the $n \times n$ identity matrix where $n = \dim(V)$.

Let β be an ordered basis for V.

Then,

$$\det(T - \lambda I) = \det([T]_\beta - \lambda I_n)$$

proof: Let β be an ordered basis for V. Then,

def. of det

$$\det(T - \lambda I) = \det([T - \lambda I]_\beta)$$

$$=$$

(next page)

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$$= \det\left([T]_\beta + [-\lambda I]_\beta\right)$$

$$= \det\left([T]_\beta - \lambda [I]_\beta\right)$$

$$= \det\left([T]_\beta - \lambda I_n\right)$$

HW 4 #2

$[T+S]_\beta = [T]_\beta + [S]_\beta$

$[cT]_\beta = c[T]_\beta$

HW 5 #2

$[I]_\beta = I_n$

$\blacksquare$

Def: Let V be a finite-dimensional vector space over a field F.
Let $T: V \to V$ be a linear transformation.
Let λ be an eigenvalue of T.

Define

$$E_\lambda(T) = \{ x \in V \mid T(x) = \lambda x \}$$
$$= N(T - \lambda I)$$

$E_\lambda(T)$ is called the <u>eigenspace of T corresponding to λ</u>.

The dimension of $E_\lambda(T)$ is called the <u>geometric multiplicity</u> of λ.

<u>Note:</u> In HW 5 you will show $E_\lambda(T)$ is a subspace of V. Also, $E_\lambda(T)$ contains $\vec{0}$ and all the eigenvectors corresponding to λ.

<u>Def:</u> Let V be a finite dimensional vector space over a field F. Let β be any ordered basis for V. Let $T: V \to V$ be a linear transformation.

The function

$$f_T(\lambda) = \det(T - \lambda I) = \det([T]_\beta - \lambda I_n)$$

is called the <u>characteristic polynomial of T</u>. The roots of $f_T(\lambda)$ are the eigenvalues of T.

If λ_0 is a root of $f_T(\lambda)$ then the <u>algebraic multiplicity</u> of λ_0 is the largest positive integer k such that $(\lambda - \lambda_0)^k$ is a factor of $f_T(\lambda)$.

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix}.$$

[pg 10]

T is a linear transformation. [You can check]

Let's find the eigenvalues, the eigenspaces, and more...

Let $\beta = [v_1, v_2, v_3]$ where

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We have

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus, $[T]_\beta = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Thus, by the previous theorem,

$$f_T(\lambda) = \det(T - \lambda I)$$

$$= \det\left([T]_\beta - \lambda I_3\right)$$

3x3 because $V = \mathbb{R}^3$ has $\dim(\mathbb{R}^3) = 3$

$$= \det\left(\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

expand on row 1

$$= \det\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -\lambda \underbrace{\begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}}_{\begin{vmatrix} \cancel{-\lambda} & \cancel{0} & \cancel{-2} \\ \cancel{1} & 2-\lambda & 1 \\ \cancel{1} & 0 & 3-\lambda \end{vmatrix}} - 0 \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix}}_{\begin{vmatrix} \cancel{-\lambda} & \cancel{0} & \cancel{-2} \\ 1 & \cancel{2-\lambda} & 1 \\ 1 & \cancel{0} & 3-\lambda \end{vmatrix}} + (-2) \underbrace{\begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix}}_{\begin{vmatrix} \cancel{-\lambda} & \cancel{0} & \cancel{-2} \\ 1 & 2-\lambda & \cancel{1} \\ 1 & 0 & \cancel{3-\lambda} \end{vmatrix}}$$

$$= -\lambda\left[(2-\lambda)(3-\lambda) - 0\right] - 0 - 2\left[0 - (2-\lambda)\right]$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

Math 4570

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pg 1

- No class this Weds
  It's a holiday

- Test 2 is next week
  on Weds the 18th

  Covers $\underbrace{\text{HW 3}}_{\text{linear trans.}}$ and $\underbrace{\text{HW 4.}}_{\text{matrix of a linear trans.}}$

- Same method for test 2 as last time. I'll email it to you on Weds morning and you send it back to me by Thursday at noon. I'll also post it on canvas.

  Send it back to me either as:

lastname.firstname.pdf

lastname_firstname.pdf

underscore or space

EX:

From last time:

$$T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad \beta = [v_1, v_2, v_3)$$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} \qquad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[T]_\beta = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

$$f_T(\lambda) = \det(T - \lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

Rational roots thm: Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where a_i are integers, $a_n \neq 0$, $a_0 \neq 0$.
If a rational number $\frac{p}{q}$ is a root of $f(x)$ then p divides a_0 and q divides a_n.

The possible rational roots $\frac{p}{q}$ of
$-\lambda^3+5\lambda^2-8\lambda+4$ must satisfy
p divides 4 and q must divide -1
So, $p=\pm 1, \pm 2, \pm 4$ and $q=\pm 1$.
Thus, the possible rational roots are
$\frac{p}{q}=\pm 1, \pm 2, \pm 4$.

Check:

$f_T(1)=-(1)^3+5(1)-8(1)+4=0$

$f_T(-1)=-(-1)^3+5(-1)-8(-1)+4=8\neq 0$

$f_T(2)=0$

$f_T(-2)\neq 0$, $f_T(\pm 4)\neq 0$

So, $\lambda=1$ and $\lambda=2$ are the only rational roots of

of $-\lambda^3+5\lambda^2-8\lambda+4$.

Since $\lambda = 2$ is a root, we know $\lambda - 2$ divides $-\lambda^3 + 5\lambda^2 - 8\lambda + 4$.

$$
\begin{array}{r|l}
 & \quad -\lambda^2 + 3\lambda - 2 \\
\hline
\lambda - 2 & -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \\
 & -(-\lambda^3 + 2\lambda^2) \\
\hline
 & \quad 3\lambda^2 - 8\lambda + 4 \\
 & -(3\lambda^2 - 6\lambda) \\
\hline
 & \quad\quad -2\lambda + 4 \\
 & \quad -(-2\lambda + 4) \\
\hline
 & \quad\quad\quad 0
\end{array}
$$

$$a\lambda^2 + b\lambda + c = a(\lambda - r_1)(\lambda - r_2)$$

r_1, r_2 are the roots

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4(-1)(-2)}}{2(-1)} = 2, 1$$

So,

$$
\begin{aligned}
-\lambda^3 + 5\lambda^2 - 8\lambda + 4 &= (\lambda - 2)(-\lambda^2 + 3\lambda - 2) \\
&= (\lambda - 2)\left[-(\lambda - 2)(\lambda - 1)\right] \\
&= -(\lambda - 1)(\lambda - 2)^2
\end{aligned}
$$

So, $-\lambda^3+5\lambda^2-8\lambda+4 = -(\lambda-1)(\lambda-2)^2$

eigenvalue of T	$\lambda=1$	$\lambda=2$
algebraic multiplicity	1	2

Recall: $E_\lambda(T) = \{x \mid T(x) = \lambda x\}$

Let's calculate $E_1(T)$

$$E_1(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -a-2c \\ a+b+c \\ a+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

So we need to solve:

$$\boxed{\begin{array}{rcl} -a -2c &=& 0 \\ a+b+c &=& 0 \\ a +2c &=& 0 \end{array}}$$

Let's solve

$$\boxed{\begin{aligned} -a \quad\;\; -2c &= 0 \\ a + b + c &= 0 \\ a \quad\;\; +2c &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\text{in reduced form}}$$

$$\boxed{\begin{aligned} a \quad\;\; +2c &= 0 \quad (1) \\ b - c &= 0 \quad (2) \\ 0 &= 0 \quad (3) \end{aligned}}$$

leading variables: a & b

free variables: c

Let $c = t$. Eq (2) gives $b = c = t$.

Eqn (1) gives $a = -2c = -2t$.

$a = -2t$
$b = t$ where t can be any real number.
$c = t$

$$E_1(T) = \left\{ \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\} \right)$$

So, a basis for $E_1(T)$ is $B_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$

So, $\dim(E_1(T)) = 1$

Thus, the geometric multiplicity of $\lambda = 1$ is $\dim(E_1(T)) = 1$.

Let's calculate

$$E_2(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is } \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

This reduces to solving:

$$\boxed{\begin{aligned} -2a \quad -2c &= 0 \\ a \quad +c &= 0 \\ a \quad +c &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boxed{\begin{aligned} a \quad +c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

leading variable: a
free variables: b, c

$$\boxed{\begin{aligned} a &= -c = -t \\ b &= s \\ c &= t \\ s,t &\in \mathbb{R} \end{aligned}}$$

$$E_2(T) = \left\{ \begin{pmatrix} -t \\ s \\ t \end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$= \left\{ t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$$

You can show, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

So, $\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$ is a basis for $E_2(T)$.

Thus, the geometric multiplicity of $\lambda = 2$ is $\dim(E_2(T)) = 2$

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Eigenvalues	$\lambda = 1$	$\lambda = 2$
algebraic mult.	1	2
geometric mult.	1	2
basis for $E_\lambda(T)$	$\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$	$\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$

Let $\beta = \beta_1 \cup \beta_2 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$

You can show that β is a linearly independent set. So, β is a basis for $\mathbb{R}^3$ since β has 3 vectors.

And,

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

So, T is diagonalizable.

Ex: $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$T(f) = f'$

$T(a+bx+cx^2) = b+2cx$

Let $\beta = [1, x, x^2]$.

We saw before that

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} f_T(\lambda) &= \det(T - \lambda I) \\ &= \det([T]_\beta - \lambda I_3) \\ &= \det\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix} \end{aligned}$$

$$\det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= \underbrace{-\lambda \begin{vmatrix} -\lambda & 2 \\ 0 & -\lambda \end{vmatrix}}_{\begin{vmatrix} \cancel{-\lambda} & \cancel{1} & \cancel{0} \\ \cancel{0} & -\lambda & 2 \\ \cancel{0} & 0 & -\lambda \end{vmatrix}} - 0 + 0$$

$$= -\lambda\left[-\lambda(-\lambda) - 2(0)\right] = -\lambda^3$$

So, $f_T(\lambda) = -\lambda^3$

The eigenvalue is $\lambda = 0$.

[ie when $-\lambda^3 = 0$]

Algebraic multiplicity of $\lambda = 0$ is 3

$$E_0(T) = \{ a+bx+cx^2 \mid \underbrace{T(a+bx+cx^2)}_{b+2cx} = 0(a+bx+cx^2) \}$$

So we need to solve

$$b + 2cx = \vec{0} = 0 + 0x + 0x^2$$

or

$$\left.\begin{aligned} b &= 0 \\ 2c &= 0 \end{aligned}\right\}$$ a can be any real number

So,

$$E_0(T) = \{ a \mid a \in \mathbb{R} \}$$
$$= \{ a \cdot 1 \mid a \in \mathbb{R} \}$$
$$= \operatorname{span}(\{1\})$$

In this case, a basis for $E_0(T)$ is $\beta = [1]$. So, $\lambda = 0$ is geometric multiplicity $\dim(E_0(T)) = 1$

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Eigenvalue	$\lambda = 0$
algebraic multiplicity	3
geometric multiplicity	1
basis for $E_\lambda(T)$	$[1]$

note:

geometric mult. of $\lambda=0$ $\leq$ alg. mult. of $\lambda=0$

There aren't enough linearly independent eigenvectors to make a basis for $V = P_2(\mathbb{R})$

So, T isn't diagonalizable.

We need 3 lin. ind. eigenvectors, we only have 1.

Math 4570

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Test 2 on Weds

Today we finish
HW 5 topic

Lemma: Let $T: V \to V$ be a linear transformation where V is a vector space over a field F. Let $v_1, v_2, \ldots, v_r$ be eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Then $v_1, v_2, \ldots, v_r$ are linearly independent.

proof: We prove this by induction on r.

Base case: Suppose $r = 1$. So, suppose v_1 is an eigenvector of T. Then, $v_1 \neq \vec{0}$. Suppose $c_1 v_1 = \vec{0}$ where $c_1 \in F$.

If $c_1 \neq 0$, then c_1^{-1} exists in F.

So, $c_1^{-1} c_1 v_1 = c_1^{-1} \vec{0}$.

Then, $v_1 = \vec{0}$ which isn't the case.

So, if $c_1 v_1 = \vec{0}$, then $c_1 = 0$.

Hence, v_1 is lin. ind.

<u>Induction hypothesis</u>: Suppose any k eigenvectors of T with distinct eigenvalues are lin. ind.

<u>Proof of k+1 case using the ind. hyp</u>:

Suppose $v_1, v_2, \dots, v_k, v_{k+1}$ are eigenvectors of T with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_{k+1}$ where $\lambda_i \neq \lambda_j$ if $i \neq j$.

Suppose

$$\boxed{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + c_{k+1} v_{k+1} = \vec{0} \quad (*)}$$

where $c_i \in F$.

Apply T to $(*)$ and use the fact that T is linear and $T(v_i) = \lambda_i v_i$ and $T(\vec{0}) = \vec{0}$ to get:

$$\boxed{\begin{aligned} c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 &+ \cdots \\ &+ c_k \lambda_k v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0} \end{aligned}} \quad (**)$$

Also, multiplying $(*)$ by λ_{k+1} gives:

$$\boxed{\begin{aligned} c_1 \lambda_{k+1} v_1 + c_2 \lambda_{k+1} v_2 &+ \cdots \\ &+ c_k \lambda_{k+1} v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0} \end{aligned}} \quad (***)$$

Computing (**)−(***) gives

$$\boxed{c_1(\lambda_1-\lambda_{k+1})v_1 + c_2(\lambda_2-\lambda_{k+1})v_2 + \dots + c_k(\lambda_k-\lambda_{k+1})v_k = \vec{0}} \quad (****)$$

Since we have k eigenvectors $v_1, \dots, v_k$ with distinct eigenvalues we can apply the ind. hyp. and thus $v_1, v_2, \dots, v_k$ are lin. ind.

So, in (****) we get

$$c_1(\lambda_1-\lambda_{k+1})=0$$
$$c_2(\lambda_2-\lambda_{k+1})=0$$
$$\vdots$$
$$c_k(\lambda_k-\lambda_{k+1})=0$$

Since, $\lambda_i-\lambda_{k+1}\neq 0$ when $1\le i\le k$ this implies that $c_1=c_2=\dots=c_k=0$.

So $(*)$ becomes

$$c_{k+1} v_{k+1} = \vec{0}.$$

As we saw in the base case, since $v_{k+1} \neq \vec{0}$ (because its an eigenvector) we must have $c_{k+1} = 0$.

Thus $c_1 = c_2 = \dots = c_k = c_{k+1} = 0$ and $v_1, v_2, \dots, v_k, v_{k+1}$ are lin. ind.

QED

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F.
Let $n = \dim(V)$.
Let $T : V \to V$ be a linear transformation.
Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigenvalues of T.
Let $n_1, n_2, \ldots, n_r$ be their geometric multiplicities,
that is $n_i = \dim(E_{\lambda_i}(T))$.

For each i, let

$$\beta_i = [v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}]$$

be an ordered basis for $E_{\lambda_i}(T)$.

Let

$$\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_r$$
$$= [v_{1,1}, v_{1,2}, \ldots, v_{1,n_1},$$
$$v_{2,1}, v_{2,2}, \ldots, v_{2,n_2},$$
$$\ldots$$
$$v_{r,1}, v_{r,2}, \ldots, v_{r,n_r}]$$

Then β is a linearly independent set.
But β might not be a basis for V.

Moreover,

β is a basis for V

iff $|\beta| = n_1 + n_2 + \cdots + n_r = n$

iff T is diagonalizable.

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<u>Proof:</u> We first show that β is a linearly independent set.

Suppose that

$$\boxed{\sum_{i=1}^{r} \sum_{k=1}^{n_i} c_{i,k} v_{i,k} = \vec{0} \qquad (*)}$$

where $c_{i,k} \in F$.

For each i, we have that $v_{i,1}, v_{i,2}, \ldots, v_{i,n_i} \in E_{\lambda_i}(T)$.

By HW 5 #6, $E_{\lambda_i}(T)$ is a subspace of V, thus

$$w_i = \sum_{k=1}^{n_i} c_{i,k} v_{i,k}$$

is in $E_{\lambda_i}(T)$.

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So, $(*)$ becomes

$$\boxed{w_1 + w_2 + \cdots + w_r = \vec{0}. \quad (**)}$$

We will now show that $w_i = \vec{0}$ for all i.

Suppose that this isn't the case.
By renumbering/reordering if necessary, there exists m with $1 \le m \le r$ such that $w_i \neq \vec{0}$ for $1 \le i \le m$ and $w_i = \vec{0}$ for $m < i$.

$$\boxed{\underbrace{w_1, w_2, \ldots, w_m}_{\text{all } \neq \vec{0}}, \underbrace{w_{m+1}, \ldots, w_r}_{\text{all } = \vec{0}}}$$

So, $(**)$ becomes

$$\boxed{w_1 + w_2 + \cdots + w_m = \vec{0}. \quad (***)}$$

But then since each w_i is $E_{\lambda_i}(T)$ and non-zero, we have m eigenvectors with distinct eigenvalues with a dependency relation $(***)$ ie, $w_1, w_2, \ldots, w_m$ are lin. dep. which contradicts the lemma.

Thus, $w_1 = w_2 = \cdots = w_r = \vec{0}$.

Ergo, $w_i = \boxed{\sum_{k=1}^{n_i} c_{i,k} v_{i,k} = \vec{0}}$

for each i. But by assumption $\beta_i = [v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}]$ is a basis, and so lin. ind., thus $c_{i,k} = 0$.

Hence β is a lin. ind. set.

$\boxed{\text{Moreover part:}}$

Note that β is a basis for V iff $|\beta| = n$ iff $n = n_1 + n_2 + \dots + n_r$.

Now we show $n = n_1 + n_2 + \dots + n_r$ iff T is diagonalizable.

($\Leftarrow$) Suppose that T is diagonalizable. This means there exists an ordered basis γ of V of eigenvectors of T.

Let $\gamma_i = \gamma \cap E_{\lambda_i}(T)$ for $i = 1, 2, \dots, r$.

So, $\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_r$.

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Then,

$$\underset{\boxed{n=\dim(V)}}{n} = \dim(\underbrace{\mathrm{span}(\gamma)}_{V}) = \sum_{i=1}^{r} \dim(\mathrm{span}(\gamma_i))$$

And

$$\dim(\underbrace{\mathrm{span}(\gamma_i)}_{\substack{\text{subspace} \\ \text{of } E_{\lambda_i}(T)}}) \le \dim(E_{\lambda_i}(T)) = n_i$$

So putting this together gives

$$\boxed{n \le \sum_{i=1}^{r} \dim(\mathrm{span}(\gamma_i)) \le n_1 + n_2 + \dots + n_r}$$

But since β is a lin. ind. set of $n_1 + n_2 + \dots + n_r$ elements inside of V which has dimension n, we must have $\boxed{n_1 + n_2 + \dots + n_r \le n}$.

So, $n = n_1 + n_2 + \dots + n_r$.

($\Leftarrow$) Suppose that

$$\underbrace{n}_{\dim(V)} = \underbrace{n_1 + n_2 + \cdots + n_r}_{\#\text{ elements in } \beta}$$

Then, β is a basis for V of eigenvectors of T. [Because we know β is lin. ind set, and if $|\beta| = \dim(V)$ it must span V also.]

Thus, T is diagonalizable. $\blacksquare$

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- Final is cumulative covers up to HW 5 (Eigenvalues, Eigenvectors, Diagonalization)

- We will talk about HW 6 material but it won't be on the final

11/30 Topic 6	12/2 Topic 6
12/7 Topic 6	12/9 Review
12/14 FINAL 12-2	

We will do same procedure as before, you pick your time window & turn it in on Tuesday noon.

One more thing with eigenvalues

Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.
Let $n = \dim(V)$.

Then:

① Let λ be an eigenvalue of T.
Let k be the algebraic multiplicity of λ. Then

$$1 \leq \dim(E_\lambda(T)) \leq k$$

[$1 \leq$ geometric mult. of λ $\leq$ alg. mult. of λ]

② T is diagonalizable iff (geometric mult. of λ) = (algebraic mult. of λ)

for all eigenvalues λ of T.

Topic 6 – Inner Product Spaces

Def: Let $z = x + iy$ be in $\mathbb{C}$.

The conjugate of z is $\overline{z} = x - iy$

The absolute value of z is $|z| = \sqrt{x^2 + y^2}$

The real part of z is $\mathrm{Re}(z) = x$

The imaginary part of z is $\mathrm{Im}(z) = y$

Ex: $\overline{5 - 3i} = 5 + 3i$

$$|5 - 3i| = \sqrt{(5)^2 + (-3)^2} = \sqrt{34}$$

$$\mathrm{Re}(10 + 13i) = 10$$

$$\mathrm{Im}(2 - 3i) = -3$$

$$(2+i)(1-3i) = 2 - 6i + i - 3i^2$$

$$= 2 - 5i + 3 = 5 - 5i$$

$i^2 = -1$

Theorem: (HW 6)

Let $z, w \in \mathbb{C}$.

Then:

① $\overline{\overline{z}} = z$

② $\overline{z+w} = \overline{z} + \overline{w}$

③ $\overline{zw} = \overline{z}\,\overline{w}$

④ $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ (if $w \neq 0$)

⑤ $z\overline{z} \in \mathbb{R}$ with $z\overline{z} \geq 0$.
Furthermore, $z\overline{z} = 0$ iff $z = 0$.

⑥ $|z|^2 = z\overline{z}$
So, $|z| = 0$ iff $z = 0$.

Def: Let V be a vector space over the field $F=\mathbb{R}$ or $F=\mathbb{C}$.
An inner product on V is a function that assigns to any ordered pair of vectors x and y in V a scalar in F, denoted by $\langle x,y\rangle$, such that the following are true for all $x,y,z\in V$ and $c\in F$:

① $\langle x+z, y\rangle = \langle x,y\rangle + \langle z,y\rangle$

② $\langle cx, y\rangle = c\langle x,y\rangle$

③ $\overline{\langle x,y\rangle} = \langle y,x\rangle$

④ $\langle x,x\rangle \in \mathbb{R}$ and if $x\neq\vec{0}$ then $\langle x,x\rangle > 0$.

We call such a V an inner product space.

Ex: Let

$V = \mathbb{R}^n$ and $F = \mathbb{R}$

or

$V = \mathbb{C}^n$ and $F = \mathbb{C}$.

Given $x, y \in V$ with

$x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$

define

$$\langle x, y \rangle = a_1 \overline{b_1} + a_2 \overline{b_2} + \cdots + a_n \overline{b_n}$$

Note if $V = \mathbb{R}^n$, then

$$\left.\begin{aligned} \langle x, y \rangle &= a_1 \overline{b_1} + a_2 \overline{b_2} + \cdots + a_n \overline{b_n} \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}\right\} \text{normal dot product}$$

because $\overline{r} = \overline{r + i0} = r - i0 = r$ if $r \in \mathbb{R}$

Ex: Using the previous example.

$V = \mathbb{C}^3$, $F = \mathbb{C}$

$$\langle (i, 1, \tfrac{1}{2}), (1+i, 0, 10) \rangle$$

$$= (i)\overline{(1+i)} + (1)\overline{(0)} + (\tfrac{1}{2})\overline{(10)}$$

$$= (i)(1-i) + (1)(0) + (\tfrac{1}{2})(10)$$

$$= i - i^2 + 0 + 5 = i + 1 + 5 = 6 + i$$

$i^2 = -1$

$V = \mathbb{R}^2$, $F = \mathbb{R}$

$$\langle (1, \pi), (-1, \tfrac{1}{2}) \rangle = (1)(-1) + (\pi)(\tfrac{1}{2})$$

$$= -1 + \frac{\pi}{2}$$

These inner products are called the <u>standard inner products</u> on $\mathbb{R}^n$ and $\mathbb{C}^n$

Let's prove that the standard inner product is actually an inner product.

Let $V = \mathbb{C}^n$, $F = \mathbb{C}$ or $V = \mathbb{R}^n$, $F = \mathbb{R}$.

Let $x, y, z \in V$ and $c \in F$.

So, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, and $z = (z_1, z_2, \ldots, z_n)$.

Then,

1

$\langle x+y, z \rangle$

$$= (x_1+y_1)\overline{z_1} + (x_2+y_2)\overline{z_2} + \ldots + (x_n+y_n)\overline{z_n}$$

$$= x_1\overline{z_1} + x_2\overline{z_2} + \ldots + x_n\overline{z_n}$$

$$+ y_1\overline{z_1} + y_2\overline{z_2} + \ldots + y_n\overline{z_n}$$

$$= \langle x, z \rangle + \langle y, z \rangle$$

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② $\langle cx, y\rangle$

$$= (cx_1)\overline{y_1} + (cx_2)\overline{y_2} + \dots + (cx_n)\overline{y_n}$$

$$= c\left[x_1\overline{y_1} + x_2\overline{y_2} + \dots + x_n\overline{y_n}\right]$$

$$= c\langle x, y\rangle$$

③ $\overline{\langle x, y\rangle} = \overline{x_1\overline{y_1} + x_2\overline{y_2} + \dots + x_n\overline{y_n}}$

$\overline{a+b} = \overline{a} + \overline{b}$

$$= \overline{x_1\overline{y_1}} + \overline{x_2\overline{y_2}} + \dots + \overline{x_n\overline{y_n}}$$

$\overline{ab} = \overline{a}\,\overline{b}$

$$= \overline{x_1}\,\overline{\overline{y_1}} + \overline{x_2}\,\overline{\overline{y_2}} + \dots + \overline{x_n}\,\overline{\overline{y_n}}$$

$\boxed{\overline{\overline{a}} = a}$

$$= \overline{x_1}\,y_1 + \overline{x_2}\,y_2 + \dots + \overline{x_n}\,y_n$$

$$= y_1\overline{x_1} + y_2\overline{x_2} + \dots + y_n\overline{x_n}$$

$$= \langle y, x\rangle$$

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(4) Note that $x_i \overline{x_i} \in \mathbb{R}$ and $x_i \overline{x_i} \geq 0$.

So,

$$\langle x, x \rangle = x_1 \overline{x_1} + x_2 \overline{x_2} + \dots + x_n \overline{x_n}$$

is also a real number and $\langle x, x \rangle \geq 0$.

If $x \neq \vec{0}$, then at least one of $x_i \neq 0$ and so $x_i \overline{x_i} > 0$

and so

$$\langle x, x \rangle = \underbrace{x_1 \overline{x_1}}_{\geq 0} + \dots + \underbrace{x_i \overline{x_i}}_{> 0} + \dots + \underbrace{x_n \overline{x_n}}_{\geq 0} > 0.$$

$\blacksquare$

If $z = a + ib$, then

$$z\overline{z} = (a+ib)(a-ib)$$
$$= a^2 - iab + iab - i^2 b^2$$
$$= a^2 + b^2 \in \mathbb{R} \text{ and } z\overline{z} \geq 0$$

<u>Theorem:</u> Let V be an inner product space over $F=\mathbb{R}$ or $F=\mathbb{C}$. Then for all $x,y,z\in V$ and $c\in F$ we have that:

① $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

② $\langle x, cy\rangle = \bar{c}\langle x,y\rangle$

③ $\langle x,x\rangle = 0$ iff $x=\vec{0}$

④ If $\langle v,y\rangle = \langle v,z\rangle$ for all $v\in V$, then $y=z$.
Similarly if $\langle y,v\rangle = \langle z,v\rangle$ for all $v\in V$, then $y=z$.

<u>pf:</u> ①, ②, ③ are in HW 6

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④ Set $v = y - z$.

Then,

$$\langle y-z, y \rangle = \langle y-z, z \rangle$$

by assumption $\langle v, y \rangle = \langle v, z \rangle$

So,

$$\langle y-z, y \rangle - \langle y-z, z \rangle = 0$$

By part 2,

$$\langle y-z, y \rangle + \langle y-z, -z \rangle = 0$$

By part 1,

$$\langle y-z, y-z \rangle = 0.$$

By part 3,

$$y - z = \vec{0}.$$

So, $y = z$. ☒

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Def: Let V be an inner product space over $F=\mathbb{R}$ or $F=\mathbb{C}$.
We say that two vectors $x,y\in V$ are <u>orthogonal</u> if $\langle x,y\rangle=0$.

We write $x\perp y$ to mean that x and y are orthogonal.

A subset $S\subseteq V$ is <u>orthogonal</u> if $x\perp y$ for all $x,y\in S$ with $x\neq y$.

Ex: Let $V=\mathbb{C}^n$, $F=\mathbb{C}$, using the standard inner product.

$$\left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}\right\rangle = (1)\overline{(0)}+(0)\overline{(3)}+(1)\overline{(0)}$$
$$=(1)(0)+(0)(3)+(1)(0)=0$$

So, $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}$ are orthogonal.

$$\left\langle \begin{pmatrix} i \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 0 \\ 10 \end{pmatrix} \right\rangle = (i)\overline{(-i)} + (2)\overline{(0)} + (1)\overline{(10)}$$ pg 2

$$= (i)(i) + (2)(0) + (1)(10)$$

$$= -1 + 0 + 10 = 9.$$

So, $\begin{pmatrix} i \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 0 \\ 10 \end{pmatrix}$ are not orthogonal.

Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 0$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = 0$$

$$\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = 0$$

So, S is an orthogonal set.

<u>Def:</u> Let V be an inner product space over $F=\mathbb{R}$ or $F=\mathbb{C}$. Given $x \in V$ we define the <u>norm</u> or <u>length</u> of x to be

$$\|x\| = \sqrt{\langle x, x \rangle}$$

<u>Ex:</u> Using the standard inner product,

if $x=(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ then

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1\overline{x_1} + x_2\overline{x_2} + \cdots + x_n\overline{x_n}}$$

if $x=(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ then

$$\|x\| = \sqrt{\langle x, x \rangle} = \underbrace{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}_{\text{usual norm you're used to in } \mathbb{R}^n}$$

Ex: Using the standard inner product we have

$$\left\|\begin{pmatrix} i \\ 1 \end{pmatrix}\right\| = \sqrt{\left\langle \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle}$$

$$= \sqrt{(i)\overline{(i)} + (1)\overline{(1)}}$$

$$= \sqrt{(i)(-i) + (1)(1)}$$

$$= \sqrt{-i^2+1} = \sqrt{2}$$

$i^2 = -1$

$$\left\|\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\| = \sqrt{\left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle}$$

$$= \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}$$

<u>Def:</u> Let V be an inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$.

A vector x is called a <u>unit vector</u> if $\|x\| = 1$.

A subset S of V is called an <u>orthonormal set</u> if

① S is an orthogonal set

and ② every vector in S is a unit vector

<u>Ex:</u> $V = \mathbb{R}^n$, $F = \mathbb{R}$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

From earlier S is an orthogonal set and

$$\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{1^2 + 0^2 + 0^2} = 1, \quad \left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| = 1$$

So, S is an orthonormal set

<u>Theorem:</u> Let V be an inner-product space over $F=\mathbb{R}$ or $F=\mathbb{C}$. If $S=\{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of non-zero vectors from V, then S is a linearly independent set.

<u>proof:</u> Suppose

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \vec{0}$$

where $c_i \in F$.

Pick v_i and inner-product both sides with v_i to get

$$\langle c_1 v_1 + \ldots + c_i v_i + \ldots + c_n v_n, v_i \rangle = \langle \vec{0}, v_i \rangle$$

This becomes

$$\langle c_1 v_1, v_i \rangle + \ldots + \langle c_i v_i, v_i \rangle + \ldots + \langle c_n v_n, v_i \rangle = 0$$

Which becomes

$$c_1\langle v_1,v_i\rangle+\dots+c_i\langle v_i,v_i\rangle+\dots+c_n\langle v_n,v_i\rangle=0$$

Since S is an orthogonal set,

$$\langle v_j,v_i\rangle=0 \quad \text{when } i\neq j.$$

Thus the above equation becomes

$$c_1\cdot 0+\dots+c_i\langle v_i,v_i\rangle+\dots+c_n\cdot 0=0$$

That is,

$$c_i\langle v_i,v_i\rangle=0$$

Since $v_i\neq\vec{0}$, $\langle v_i,v_i\rangle\neq 0$,

so $c_i=0$.

So, $c_1=c_2=\dots=c_n=0$ and S is a lin. ind. set. $\blacksquare$

Theorem: Let V be an inner product space over $F=\mathbb{R}$ or $F=\mathbb{C}$. Let $\beta=\{v_1, v_2, \ldots, v_n\}$ be a basis for V. Let $x \in V$.

① If β is an orthogonal set, then

$$x = \frac{\langle x, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle x, v_2\rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle x, v_n\rangle}{\|v_n\|^2} v_n$$

② If β is an orthonormal set, then

$$x = \langle x, v_1\rangle v_1 + \langle x, v_2\rangle v_2 + \cdots + \langle x, v_n\rangle v_n$$

pf: We just need to prove 1.

Suppose β is an orthogonal basis for V.

Let $x \in V$.

Then since β spans V, we can write

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_i \in F$.

Pick a v_i and inner product both sides with v_i to get

$$\begin{aligned}
\langle x, v_i \rangle &= \langle c_1 v_1 + \dots + c_n v_n, v_i \rangle \\
&= \langle c_1 v_1, v_i \rangle + \dots + \langle c_i v_i, v_i \rangle + \dots + \langle c_n v_n, v_i \rangle \\
&= c_1 \underbrace{\langle v_1, v_i \rangle}_{0} + \dots + c_i \langle v_i, v_i \rangle + \dots + c_n \underbrace{\langle v_n, v_i \rangle}_{0}
\end{aligned}$$

S is an orthogonal set

$$= c_i \langle v_i, v_i \rangle$$

Solve for c_i to get

$$c_i = \frac{\langle x, v_i \rangle}{\langle v_i, v_i \rangle} = \frac{\langle x, v_i \rangle}{\left(\sqrt{\langle v_i, v_i \rangle}\right)^2} = \frac{\langle x, v_i \rangle}{\|v_i\|^2} \quad \blacksquare$$

Gram - Schmidt Process

Let V be an inner product space over $F=\mathbb{R}$ or $F=\mathbb{C}$.
Let $S=\{w_1, w_2, \ldots, w_n\}$ be a linearly independent subset of V.
Define $S'=\{v_1, v_2, \ldots, v_n\}$ as follows:

$$v_1 = w_1$$

$$v_k = w_k - \underbrace{\sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j}_{\text{this is the projection of } w_k \text{ onto } \operatorname{span}(\{v_1, v_2, \ldots, v_{k-1}\})} , \quad 2 \le k \le n$$

Then S' is an orthogonal set of linearly independent vectors where $\operatorname{span}(S) = \operatorname{span}(S')$.

Therefore, if S above is a basis for V, then given S' as above we can construct

$$S'' = \left\{ \frac{1}{\|v_1\|} v_1, \ldots, \frac{1}{\|v_n\|} v_n \right\}$$

And S'' will be an orthonormal basis for V.

Hence every finite-dimensional inner-product space V over $F = \mathbb{R}$ or $F = \mathbb{C}$ has basis.
an orthonormal

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More on eigenvectors

<u>Theorem:</u> Let V be a finite-dimensional inner-product space over $F = \mathbb{C}$ or $F = \mathbb{R}$. Let $T : V \to V$ be a linear transformation. There exists a unique function $T^* : V \to V$ such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

for all $x, y \in V$. Furthermore, T^* is a linear transformation.

T^* is called the <u>adjoint</u> of T.

Ex: $V = \mathbb{C}^3$, $F = \mathbb{R}$

Use the standard inner-product.

$T: \mathbb{C}^3 \to \mathbb{C}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2b+ic \\ ia \\ b \end{pmatrix}$

T is a linear transformation.

Let's find T^*.

Then,

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, T^*\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle = \left\langle T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} 2x_2+ix_3 \\ ix_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle$$

$$= (2x_2+ix_3)\overline{y_1} + (ix_1)\overline{y_2} + x_2\overline{y_3}$$

$$= 2x_2\overline{y_1} + ix_3\overline{y_1} + ix_1\overline{y_2} + x_2\overline{y_3}$$

$$= x_1(i\overline{y_2}) + x_2(2\overline{y_1}+\overline{y_3}) + x_3(i\overline{y_1})$$

$$= x_1(\overline{-i}\,\overline{y_2}) + x_2(\overline{2y_1+y_3}) + x_3(\overline{-i}\,\overline{y_1})$$

$$= x_1(\overline{-iy_2}) + x_2(\overline{2y_1+y_3}) + x_3(\overline{-iy_1})$$

$$= \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} -iy_2 \\ 2y_1 + y_3 \\ -iy_1 \end{pmatrix} \right\rangle$$

pg 3

Since T^* is unique,

$$\boxed{T^* \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -iy_2 \\ 2y_1 + y_3 \\ -iy_1 \end{pmatrix}}$$

If $\beta = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$ is the standard basis for $\mathbb{C}^3$ then

$$[T]_\beta = \begin{pmatrix} 0 & 2 & i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$[T^*]_\beta = \begin{pmatrix} 0 & -i & 0 \\ 2 & 0 & 1 \\ -i & 0 & 0 \end{pmatrix}$$

You can calculate these

$[T^*]_\beta$ is gotten from $[T]_\beta$ by transposing and conjugating the elements in the matrix $[T]_\beta$

<u>Def:</u> Let A be an $n \times n$ matrix with entries from $\mathbb{C}$. Then A^* is <u>defined</u> by

$$(A^*)_{i,j} = \overline{(A_{j,i})}$$

That is, A^* is obtained by transposing A and conjugating the elements.

<u>Ex:</u>

$$\begin{pmatrix} 1+i & 3 \\ i & \frac{1}{2}+2i \end{pmatrix}^* = \begin{pmatrix} 1-i & -i \\ 3 & \frac{1}{2}-2i \end{pmatrix}$$

<u>Theorem:</u> Let V be a finite-dimensional inner product space over $F=\mathbb{C}$ or $F=\mathbb{R}$.

Let β be an ordered orthonormal basis for V.

Let $T: V \to V$ be a linear transformation.

Then,

$$[T^*]_\beta = \left([T]_\beta\right)^*$$

<u>Def:</u> Let V be a finite-dimensional inner product space over $F = \mathbb{C}$ or $F = \mathbb{R}$.
Let $T : V \to V$ be a linear transformation

(a) We say that T is <u>normal</u> if $\underbrace{TT^* = T^*T}_{(T \circ T^*)(x) = (T^* \circ T)(x) \text{ for all } x \in V}$.

(b) We say that T is <u>self-adjoint</u> or <u>Hermitian</u> if $T = T^*$.

<u>Note:</u> If T is self-adjoint, then T is normal.

Theorem: Let V be a finite-dimensional inner product space over $F = \mathbb{C}$. Let $T : V \to V$ be a linear transformation. Then T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

Theorem: Let V be a finite-dimensional inner product space over $F = \mathbb{R}$. Let $T : V \to V$ be a linear transformation. Then T is self-adjoint if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

Spectral Theorem for symmetric real matrices

Suppose that A is an $n \times n$ symmetric matrix $(A^T = A)$ with real entries. Then:

① All the eigenvalues of A are real.

② There is an orthonormal basis for $\mathbb{C}^n$ consisting of n real eigenvectors of A.

③ If λ is an eigenvalue of A, then the alg. mult. of A equals the geometric mult. of A.

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Final on Monday 12/14. $\boxed{12-2}$

Final is cumulative

Covers up to HW 5

[HW 6 is <u>not</u> on the final]

I'll email you the final by 8am on Monday. I'll try to make it appear on canvas at 5am.

Use up to $2\frac{1}{2}$ hours.

Email it back to me by Tuesday 12/15 at noon.

---

lastname. firstname. pdf

lastname_firstname. pdf

pg 2

Test 1 – 90%

Test 2 – 70%

Final 3 – 80%

<u>syllabus:</u>

$$\frac{1}{3}90\% + \frac{1}{3}70\% + \frac{1}{3}80\% = 80\%$$

<u>new method:</u>

$$\frac{1}{2}90\% + \frac{1}{2}80\% = 85\%$$

HW 5

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① (c) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+b \\ 3b \\ 4c \end{pmatrix}$$

(i) $\beta = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[T]_\beta = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$f_T(\lambda) = \det\left([T]_\beta - \lambda I_3\right) =$$

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$$= \det\left(\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} + 0 + 0$$

$$= (3-\lambda)\left[(3-\lambda)(4-\lambda) - 0\right]$$

$$= (3-\lambda)^2(4-\lambda)$$

eigenvalue	alg. mult.
$\lambda = 3$	2
$\lambda = 4$	1

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$$E_3(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} 3a+b \\ 3b \\ 4c \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \\ 3c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} b \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{\begin{aligned} b &= 0 \\ 0 &= 0 \\ c &= 0 \end{aligned}}$$

$$E_3(T) = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \right)$$

So, $\beta_1 = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$ is a basis for $E_3(T)$. So, geometric mult of $\lambda = 3$ is 1.

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$$E_4(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 4\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} 3a+b \\ 3b \\ 4c \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \\ 4c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \begin{pmatrix} -a+b \\ -b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\boxed{\begin{array}{rl} -a+b & = 0 \\ -b & = 0 \\ 0 & = 0 \end{array}}$$

$\leftarrow b = 0$

$a = b = 0.$

c can be any real #

$$E_4(T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

$$= \left\{ c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\} = \operatorname{span}\left(\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}\right)$$

basis for $E_4(T)$ is $\beta_2 = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$

geometric mult. of $\lambda = 4$ is 1.

λ	alg. mult of λ	basis for $E_\lambda(T)$	geometric mult of λ
3	2	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	1
4	1	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	1

T is <u>not</u> diagonalizable since (alg. mult. of $\lambda = 3$) $\neq$ (geom. mult. of $\lambda = 3$)

Not enough eigenvectors to diagonalize

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(3) $A \in M_{n,n}(F)$ is diagonalizable if $\exists Q \in M_{n,n}(F)$ where Q^{-1} exists such that $Q^{-1}AQ = D$ where d is diagonal.

(a) $T: V \to V$, V finite-dim. β is an ordered basis of V

T is diagonalizable iff $[T]_\beta$ is diagonalizable

($\Longrightarrow$) Suppose T is diagonalizable. Then there exists an ordered basis γ of eigenvectors of T, where $[T]_\gamma$ is diagonal. Let $Q = [I]_\gamma^\beta$ be the change of basis matrix from γ to β.

Then, $Q^{-1}[T]_\beta Q = [I]_\beta^\gamma [T]_\beta [I]_\gamma^\beta = [T]_\gamma$

so, $[T]_\beta$ is invertible.

($\Leftarrow$) Suppose $[T]_\beta$ is diagonalizable. pg 9

Then there exists a matrix Q where Q^{-1} exists and

$$Q^{-1}[T]_\beta Q = D$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

is a diagonal matrix.

Let $\beta = [v_1, v_2, \ldots, v_n]$

Let $Q = \left(c_1 \,\middle|\, c_2 \,\middle|\, \cdots \,\middle|\, c_n \right)$

where c_i is the ith column of Q.

We have $Q^{-1}[T]_\beta Q = D$.

So, $[T]_\beta Q = QD$.

We have

$$[T]_\beta Q = \begin{array}{cccc} \left([T]_\beta c_1 \right. & \big| \; [T]_\beta c_2 & \big| \; \cdots \; \big| & \left. [T]_\beta c_n \right) \\ \text{1st column} & \text{2nd column} & & \text{last column} \end{array}$$

and

$$QD = \left(c_1 \,\middle|\, c_2 \,\middle|\, \cdots \,\middle|\, c_n \right) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$= \left(\lambda_1 c_1 \,\middle|\, \lambda_2 c_2 \,\middle|\, \cdots \,\middle|\, \lambda_n c_n \right)$$

So, $[T]_\beta c_i = \lambda_i c_i$.

Since Q^{-1} exists, none of Q's columns are $\vec{0}$. So, $c_i \neq \vec{0}$ $\forall i$.

So, c_i is an eigenvector of $[T]_\beta$ with eigenvalue λ_i.

Suppose $c_i = \begin{pmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{pmatrix}$.

Define

$$m_i = q_{1i} v_1 + q_{2i} v_2 + \cdots + q_{ni} v_n$$

so that

$$[m_i]_\beta = c_i.$$

$\boxed{\begin{array}{c} m_i \in V \\ T: V \to V \end{array}}$

Then,

$$[T(m_i)]_\beta = [T]_\beta [m_i]_\beta = [T]_\beta c_i = \lambda_i c_i$$

$$= \lambda_i \begin{pmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{pmatrix} = \begin{pmatrix} \lambda_i q_{1i} \\ \lambda_i q_{2i} \\ \vdots \\ \lambda_i q_{ni} \end{pmatrix}$$

So,

$$T(m_i) = (\lambda_i q_{1i}) v_1 + (\lambda_i q_{2i}) v_2 + \dots + (\lambda_i q_{ni}) v_n$$

$$= \lambda_i (q_{1i} v_1 + q_{2i} v_2 + \dots + q_{ni} v_n)$$

$$= \lambda_i m_i$$

Since $[m_i]_\beta = c_i \neq \vec{0}$, $m_i \neq \vec{0}$.

So, $m_1, m_2, \dots, m_n$ are eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

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$n = \dim(V)$.

We just need to show
$m_1, m_2, \dots, m_n$ are lin. ind.
Then we will have a basis of V
of eigenvectors of T.

Suppose
$$\alpha_1 m_1 + \alpha_2 m_2 + \dots + \alpha_n m_n = \vec{0}.$$

Then
$$\alpha_1 (q_{11} V_1 + q_{21} V_2 + \dots + q_{n1} V_n)$$
$$+ \alpha_2 (q_{12} V_1 + q_{22} V_2 + \dots + q_{n2} V_n)$$
$$+ \dots +$$
$$+ \alpha_n (q_{1n} V_1 + q_{2n} V_2 + \dots + q_{nn} V_n) = \vec{0}$$

So,

$$(\alpha_1 q_{11} + \dots + \alpha_n q_{1n}) v_1 + \dots + (\alpha_1 q_{n1} + \alpha_2 q_{n2} + \dots + \alpha_n q_{nn}) v_n = 0$$

Since $v_1, v_2, \dots, v_n$ is a basis,

$$\begin{aligned}
\alpha_1 q_{11} + \dots + \alpha_n q_{1n} &= 0 \\
\alpha_1 q_{21} + \dots + \alpha_n q_{2n} &= 0 \\
&\vdots \\
\alpha_1 q_{n1} + \dots + \alpha_n q_{nn} &= 0
\end{aligned}$$

$$\underbrace{\begin{pmatrix} q_{11} & \cdots & q_{1n} \\ q_{21} & \cdots & q_{2n} \\ \vdots & & \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}}_{Q} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So, $Q^{-1} Q \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

So, $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

So, $m_1, \dots, m_n$ are lin. ind. $\blacksquare$