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Abstract

Purpose This paper aims to establish that a certain sort of mathematical pluralism is true.
Methods The paper proceeds by arguing that that the best versions of mathematical Platonism and anti-Platonism both entail the relevant sort of mathematical pluralism.
Result and Conclusion This argument gives us the result that mathematical pluralism is true, and it also gives us the perhaps surprising result that mathematical Platonism and mathematical pluralism are perfectly compatible with one another.

Keywords Mathematical platonism · Mathematical pluralism · Mathematical relativism · Mathematical fictionalism

Introduction

In this paper, I will do two things: I will argue that a certain sort of mathematical pluralism, or relativism, is true; and I will argue that, perhaps surprisingly, this view is perfectly consistent with mathematical platonism.

Mathematical platonism is the view that (a) there exist abstract mathematical objects—objects that are non-spatiotemporal and wholly non-physical and non-mental—and (b) our mathematical theories provide true descriptions of such objects. So, for instance, on the platonist view, the sentence '3 is prime' says something true about a certain object—namely, the number 3—and on this view, 3 is an abstract object; i.e., it is a real and objective thing that exists independently of

✉ Mark Balaguer
mbalagu@exchange.calstatela.edu

1 Department of Philosophy, California State University, Los Angeles, CA, USA

us and our thinking, outside of space and time, and it is wholly non-physical, non-mental, and causally inert.[1]

'Mathematical pluralism,' on the other hand, is harder to define, and it has been used to denote many different views.[2] The kind of pluralism that I want to talk about is based on the loose idea that there is not just one mathematical truth—that there are many different mathematical "truths." Now, as it stands, this is obvious. Since '3 is prime' and '7 is prime' are both true, this already gives us the result that there are many mathematical truths. But, of course, this is not what pluralists have in mind. Their idea, put a bit less roughly, is that there can be different theories of the same kind, that seem to be competitors, or incompatible, that are both true. So, for instance, there can be Euclidean and non-Euclidean geometries that are both true; and there can be standard and nonstandard theories of arithmetic (that pick out standard and nonstandard models, respectively) that are both true; and there can be different set theories, containing incompatible sets of axioms, that are both true.

This sort of pluralism goes hand-in-hand with *mathematical relativism*, which says that different cultures can endorse different mathematical theories, that seem to be incompatible with one another, that are both true. So, for instance, Martians might endorse a set theory in which the continuum hypothesis (CH) is true, and we might endorse a set theory in which CH is false, and we could both be right. Put roughly, the idea is that CH would be "true for them," but not for us.

So defined, it might seem that mathematical pluralism is incompatible with mathematical platonism; it might seem that these two views are incompatible for the same reason that moral realism and moral relativism are incompatible. But I think this is false. I think that when the platonist view is properly developed, it has pluralistic consequences. However, it does not have mad-dog, foaming-at-the-mouth pluralistic consequences. The pluralistic consequences are principled and contained.

In the next three sections, I will explain exactly how and why platonists should be pluralists and relativists. Then after that, I will argue that anti-platonists should endorse the very same kind of pluralism/relativism, and so we will have an argument for the claim that this sort of pluralism/relativism is true.

Plenitudinous Platonism

As I defined it above, mathematical pluralism is pluralism about mathematical *truth*. Thus, in order to determine whether platonists should be pluralists (and if they should, what sort of pluralism they should endorse), we need to know what they think mathematical truth consists in. But in order to figure this out—in order to zero in on the best platonist theory of mathematical truth—we first need to decide whether platonists should endorse *plenitudinous platonism* or *sparse platonism*. According to plenitudinous platonism (or as I have called it elsewhere (1998), *full-*

[1] Platonism has been endorsed by numerous people, including Frege (1884), Gödel (1964), and Russell (1912).

[2] See, for example, Davies (2005), Kollner (2009), Priest (2013), and Friend (2013).

blooded platonism, or for short, FBP), the mathematical realm is plenitudinous, so that there are as many abstract mathematical objects as there could be—i.e., there actually exist abstract mathematical objects of all possible kinds. According to sparse platonism, on the other hand, the mathematical realm is not plenitudinous, so that of all the different kinds of mathematical objects that might have been instantiated, only some of them actually are instantiated. I have argued at length elsewhere (1995, 1998, 2016) for the claim that FBP is superior to any version of sparse platonism and, indeed, that it is the only tenable version of platonism. There are numerous compelling arguments for this claim, but probably the best one is based on the fact that, unlike sparse platonism, FBP can be given an acceptable epistemology. More precisely, FBP gives platonists a way of explaining how naturalistic human beings can acquire knowledge of abstract objects like numbers, despite the fact that we do not have any information-gathering contact with such objects. In a nutshell, the explanation proceeds as follows:

> Since FBP says that there are abstract mathematical objects of all possible kinds, it follows that if FBP is true, then every purely mathematical theory that *could* be true—i.e., that is internally consistent—accurately describes some collection of actually existing abstract objects. Thus, it follows from FBP that in order to acquire knowledge of abstract objects, all we have to do is come up with an internally consistent purely mathematical theory (and know that it is internally consistent). This is because, again, according to FBP, *every* consistent purely mathematical theory accurately describes a collection of actually existing abstract objects. But if all we need to do in order to acquire knowledge of abstract objects is come up with a consistent mathematical theory (and know that it is consistent), then it seems that we *can* acquire such knowledge. For it seems clear that (a) we *are* capable of formulating internally consistent mathematical theories (and of knowing that they are internally consistent), and (b) being able to do this does not require us to have any sort of information-gathering contact with the abstract objects that the theories in question are about. Thus, if all of this is right, then FBP gives platonists a way of explaining how naturalistic creatures like us could acquire knowledge of abstract objects, despite the fact that they do not have any information-gathering contact with such objects.

Another way to put this is to say that FBP gives us a sort of *recipe* for acquiring knowledge of abstract objects. The recipe goes like this: (i) Dream up a mathematical story. Or more precisely: come up with a kind of mathematical object and articulate a theory about those objects. For instance, the axioms of Peano Arithmetic (PA) give us a theory of the natural numbers, and Zermelo-Frankel set theory (ZF) gives us a theory of sets. (ii) Do your best to make sure that the theory is internally consistent. (It can often be very difficult to know that a theory is consistent, but in general, it is not impossible for us to acquire such knowledge; in particular, knowledge of the consistency of a theory does *not* require any sort of access to the objects that the theory is about.) (iii) Conclude that the theory accurately describes a collection of abstract objects; e.g., PA accurately describes a sequence of numbers; ZF accurately describes a hierarchy of sets; and so on. (iv) If you like, you can use logic to deduce further facts about the objects in question by proving theorems from the axioms you have articulated. (One nice thing about this recipe is that it is not revisionistic; it fits well with real mathematical methodology—i.e., with the methodology of laying down axioms and proving theorems.)

You might object that in order for humans to acquire knowledge of abstract objects in this way, they would first need to know that FBP is true. But this objection involves a misunderstanding of the epistemological challenge to platonism—i.e., the Benacerraf (1973) challenge that FBP-ists are trying to answer. The challenge is not a Cartesian skeptical challenge. The challenge is to explain *how* human beings could acquire knowledge of abstract objects. This challenge is easy to answer in the case of physical objects—we can do it by simply appealing to sense perception. Even if there is no good response to the Cartesian skeptical worry about our knowledge of physical objects, it is very easy to explain how humans could acquire knowledge of physical objects—they could do this by pointing their eyes at physical objects (because it could be that photons bounce off of physical objects and into the eyes of humans, carrying information about what those objects are like). This explanation is satisfying because it appeals to our *ordinary* way of acquiring beliefs about physical objects; in other words, it explains how our ordinary method of belief acquisition (namely, sense perception) could be accurate. The purpose of the FBP-ist epistemology alluded to above is to provide something analogous to the appeal to sense perception. The goal is to explain how we could acquire knowledge of abstract objects, using our *ordinary* mathematical methodology. And, again, the claim is that we could do this by (a) conceiving of a mathematical structure (or a part of the mathematical realm); (b) formulating a set of axioms that characterize the structure (or the part of the mathematical realm) that we have in mind; and (c) proving theorems from those axioms. This gives us an account of how we could acquire knowledge of abstract objects that is analogous to our perception-based account of how we could acquire knowledge of physical objects. Now, of course, you might still have a skeptical worry about mathematics; you might still wonder how we could know that there even *is* a mathematical realm. But that is a different worry. And, of course, it is a worry that applies to our ordinary knowledge of physical objects as well as to our knowledge of abstract objects, because it is not clear that we could know that there is an external world.

In order to fully develop this argument for the claim that FBP is superior to sparse platonism, I would need to argue that sparse platonism cannot be given an acceptable epistemology (i.e., that sparse platonism is incompatible with the fact that human beings are capable of acquiring mathematical knowledge), and I would need to further develop and defend the above FBP-ist epistemology. I have done all of this elsewhere (1998), but unfortunately, I do not have the space to fully develop the argument here. Nonetheless, I will be assuming in what follows that platonists have good reason to reject sparse platonism and endorse plenitudinous platonism, or FBP. At any rate, the platonist view I will develop here is a version of FBP.

(While I will not be directly developing the epistemology-based argument for the superiority of FBP over sparse platonism, a good deal of the picture will emerge in what follows; e.g., we will encounter reasons to think that sparse platonism cannot be given an acceptable epistemology, and we will see how FBP-ists can respond to various worries about their view, e.g., the worry that FBP entails a contradiction.)

Platonistic Mathematcial Truth

Let us move on now to the question of what platonists should say about mathematical truth. We can get at the main issue here by reflecting on undecidable sentences like CH. The problem is that there are structures in which CH is true and structures in which $\sim$CH is true, and so we seem to get the result that CH is true of some parts of the mathematical realm, and $\sim$CH is true of others. Indeed, we get this result even if we focus just on structures that satisfy ZFC (ZF plus the axiom of choice); for there are structures in which ZFC + CH is true and there are structures in which ZFC + $\sim$CH is true. But, given this, one might wonder what platonists should say about whether CH or $\sim$CH is true. One thing they might say here is this:

> *Silly Platonism*: A mathematical sentence or theory is true just in case it accurately characterizes some collection of mathematical objects. Thus, since the mathematical realm is plenitudinous, it follows that all consistent mathematical theories are true. And so it follows that CH and $\sim$CH are both true, because CH is true of some parts of the mathematical realm, and $\sim$CH is true of others.

But platonists do not need to say this, and for a variety of reasons (not the least of which being that Silly Platonism is straightforwardly inconsistent), they would be wise not to. Here is a better view:

> *Better Platonism*: There is a difference between being true in some particular structure and being true *simpliciter*. To be true *simpliciter*, a pure mathematical sentence needs to be true in the *intended* structure, or the intended part of the mathematical realm—i.e., the part of the mathematical realm that we have in mind in the given branch of mathematics.

This makes a good deal of sense; on this view, an arithmetic sentence is true iff it is true *of the sequence of natural numbers*; and a set-theoretic sentence is true iff it is true *of the universe of sets*—i.e., the universe of things that correspond to our intentions concerning the word 'set'—and so on. But this cannot be the whole story, for we cannot assume that there is a unique intended structure for every branch of mathematics. It may be that in some branches of mathematics, our intentions have some imprecision in them; in other words, it may be that our *full* conception of the objects being studied is not strong enough, or precise enough, to zero in on a unique structure up to isomorphism. Indeed, this might be the case in set theory. For instance, it may be that there is a pair of structures, call them H1 and H2, such that (i) ZFC + CH is true in H1, and (ii) ZFC + $\sim$CH is true in H2, and (iii) H1 and H2 both count as intended structures for set theory, because they are both perfectly consistent with *all* of our set-theoretic intentions—or with *our full conception of the universe of sets* (FCUS). Thus, the conclusion here seems to be that there could be some mathematical sentences (and it *may* be that CH is such a sentence) that are true in some intended parts of the mathematical realm and false in others. And this, of course, raises a problem for Better Platonism.

But while Better Platonism is problematic, I think it is on the right track. In particular, I like the idea of defining mathematical truth in terms of truth in intended structures, or intended parts of the mathematical realm. But platonists need to develop this idea in a way that is consistent with the fact that there can be multiple intended structures in a given branch of mathematics. The first thing they need to do here is indicate what determines whether a given structure counts as intended in a given domain. Not surprisingly, this has to do with whether the structure fits with our intentions. We can think of our intentions in a given branch of mathematics as being captured by the *full conception* that we have of the objects, or purported objects, in that branch of mathematics. Below, I will say a bit about what our various full conceptions—or as I will call them, FCs—consist in. But before I do this, I want to indicate how platonists can use our FCs to give a theory of what determines whether a structure counts as intended. They can do this by saying something like the following:

> A part P of the mathematical realm counts as intended in a branch B of mathematics iff all the sentences that are built into the FC in B—i.e., the full conception that we have of the purported objects in B—are true in P. (Actually, this isn't quite right, because it assumes that all of our FCs are consistent. But for the sake of simplicity, we can ignore this complication and work with the assumption that our FCs *are* consistent. By proceeding in this way, I won't be begging any important questions, because nothing important will depend here on what determines which structures count as intended in cases in which our FCs are inconsistent.[3])

To make this more precise, I need to clarify what our various full conceptions, or FCs, consist in. We can think of each FC as a bunch of sentences. To see which sentences are included in our FCs, we need to distinguish two different kinds of cases, namely, (a) cases in which mathematicians work with an axiom system, and there is nothing behind that system, i.e., we do not have any substantive pretheoretic conception of the objects (or purported objects) being studied; and (b) cases in

[3] Two points. First, there is nothing implausible about the idea that there could be a clear intended structure (or set of structures) in a setting in which the relevant FC was inconsistent. We might have a unique structure in mind (up to isomorphism) but have inconsistent thoughts about it. Second, let me say a few words about how platonists might proceed, if we dropped the simplifying assumption that our FCs are all consistent. They could say something of the following form:

> A part P of the mathematical realm counts as intended in a branch B of mathematics iff either (a) the relevant FC is consistent and all the sentences in the FC are true in P, or else (b) the relevant FC is inconsistent and The trick is to figure out what to plug in for the three dots. We might try something like this: "for all of the intuitively and theoretically attractive ways of eliminating the contradiction from the relevant FC, if we eliminated the contradiction in the given way, then P would come out intended by clause (a)." Or we might use a radically different approach; e.g., we might try to figure out some way to pare down the given FC—i.e., the sum total of *all* of our thoughts about the relevant objects—and zero in on a (consistent) subset of these thoughts that picked out the structure(s) that we had in mind. There are a lot of different ideas one might try to develop here, but I will not pursue this any further.

which we *do* have an intuitive, pretheoretic conception of the (purported) objects being studied. To be more precise, what I really have in mind here is a distinction between (a) cases in which any structure that satisfies the relevant axioms is *thereby* an intended structure (or a structure of the kind that the given theory is supposed to be *about*, or some such thing); and (b) cases in which a structure S could satisfy the relevant axiom system but still fail to be an intended structure (or a structure of the kind that the given theory was supposed to be about) because the axiom system failed to zero in on the kind of structure we had in mind intuitively and because S did not fit with our intuitive or pretheoretic conception. (I suppose some people might argue that, in fact, only one of these kinds of theorizing actually goes on in mathematics. This will not matter here at all, but for the record, I think this is pretty clearly false; it seems pretty obvious that arithmetic fits into category (b) and that, say, group theory fits into category (a).[4] But again, this is not going to matter here. I am going to discuss both kinds of cases, and if it turns out that only one of them is actualized in mathematical practice, that will not undermine what I say—it will simply mean that some of what I say will be unnecessary.)

In cases in which we do not have any substantive pretheoretic conception of the (purported) objects being studied, the so-called full conception of these objects, or the FC, is essentially exhausted by the given axiom system, so that any structure that satisfies the axioms is thereby an intended structure. And this makes a good deal of sense, for in cases like this, our intention just *is* to be studying the kinds of structures picked out by the given axiom system. Or, put differently, in cases of this kind, we can think of the axiom system in question as implicitly defining the kinds of objects being studied. So in cases of this kind, the notion of a full conception does not do any substantive theoretical work (and neither does the notion of an intended structure).

Things are different in cases where we do have an intuitive, pretheoretic conception of the (purported) objects being studied. I will discuss this kind of case by focusing on our full conception of the natural numbers—or FCNN. We can take FCNN to consist of a bunch of sentences, where a sentence is part of FCNN iff (roughly) either (a) it says something about the natural numbers that we (implicitly or explicitly) accept or (b) it follows from something we accept about the natural numbers.[5] This is a bit rough and simplified, so let me clarify a few points.

First, in speaking of sentences that "we accept," what I mean are sentences that are uncontroversial among mathematicians (you might want to include ordinary folk here as well, but I think it is probably better not to). This does not mean that a sentence has to be universally accepted by mathematicians in order to be part of FCNN; it just needs to be uncontroversial in the ordinary sense of the term.

Second, it is important to note that when we move away from arithmetic to other branches of mathematics, we might need to replace the word 'we' with a narrower term. If we wanted to give a general formulation here, we might say that our FCs are

[4] To appreciate this, notice that (i) nonstandard models of first-order arithmetical theories are clearly unintended models (i.e., we could not plausibly take any nonstandard model to be the sequence of natural numbers); and (ii) any structure that satisfies the axioms of group theory is *thereby* a group.

[5] The relevant notion of consequence, or following from, can be thought of as a primitive notion. Or alternatively, we can take *possibility* as a primitive and define consequence in terms of it.

determined by what is accepted by the *relevant people*, or some such thing, where a person counts as "relevant" in a given domain only if he or she is (a) sufficiently informed on the topic to have opinions that matter, and (b) accepting of the theorizing and the purported objects in question. The purpose of clause (b) is to rule out dissenters—i.e., experts who object to the very idea of the objects or the theorizing in question. The opinions and intentions of these people are not needed to determine what the intended objects are, and there is a good reason for not including them here. The reason is that I am characterizing our FCs in terms of what the relevant people *accept*, and dissenters will very likely not accept the relevant sentences at all; thus, they should not be included among the "relevant people."[6]

Third, in the end, we might want to say that in order for a sentence to be part of FCNN, it needs to be the case that we *nonspeculatively* accept it. To see why, suppose that nearly all mathematicians came to believe the twin prime conjecture, but suppose that (because they did not have a proof) they considered this belief to be speculative. In this case, even if the twin prime conjecture was almost universally accepted, we would presumably not want to say that it was part of FCNN (unless it followed from other sentences in FCNN). For if the twin prime conjecture was actually false, and if mathematicians speculatively accepted it, then the set of sentences about the natural numbers that we accepted would be inconsistent, but intuitively, our *conception* of the natural numbers would not be. And this is why we might want to require that a sentence be *nonspeculatively* accepted in order to be part of FCNN. (I say we *might* want to require this because there might be other ways to solve this problem. For instance, one might argue that nonspeculativeness is already built into the definition of 'uncontroversial,' and if so, we would not need an additional requirement here.)

Finally, I characterize FCNN in terms of the sentences we *accept*, instead of the ones we believe, for a reason: mathematicians might not literally believe the sentences in FCNN at all. Suppose, for instance, that all mathematicians suddenly became error-theoretic fictionalists[7] (or suppose the community of mathematicians was split between platonists and fictionalists); then mathematicians (or at least some of them) would not *believe* the sentences in FCNN. But they would still accept those sentences in the sense I have in mind. We do not need to get very precise about what exactly 'accept' means here, but we can at least say that various kinds of mental states will count as kinds of acceptances. For instance, a person P will count as accepting a sentence S if P is in a mental state M that counts as a belief that S is true,

[6] There could be cases where there was only one relevant person; I can theorize about a given mathematical structure even if no one else does. There could also be cases where there *seemed* to be a community but really was not. E.g., suppose that (i) two people, A and B, claimed to be talking about objects of the same kind and believed there were large discrepancies in what they accepted about these objects, and (ii) A and B were really just thinking of two different kinds of objects or structures. In this case, there would simply be two different FCs and two different kinds of intended structures.

[7] I define this view below. In a nutshell, it is the view that while our mathematical theories are supposed to be about abstract objects, there are no such things as abstract objects, and so our mathematical theories are not strictly speaking true.

or a belief that it is fictionalistically correct,[8] or a belief that it is correct in some other appropriate nominalistic sense, or a belief that it is true-or-correct, or if M is such that there is no fact of the matter whether it is a belief that S is true or a belief that it is correct, and so on. Various other kinds of mental states might also count as kinds of acceptances of S, but there is no need to list them all here.[9]

So given all this, what sentences are built into FCNN? Well, for starters, I think we can safely assume that all of the axioms of standard arithmetical theories—sentences like '0 is a number' and 'Every number has a successor'—are part of FCNN, because they are all uncontroversial in the ordinary sense of the term. Thus, the theorems of our arithmetical theories—sentences like '7 + 5 = 12' and 'There are infinitely many primes'—are also part of FCNN. (The same goes for our full conception of the universe of sets (FCUS), and this brings out an important point, namely, that our FCs are not wholly pretheoretic or intuitive—they are theoretically informed. It is implausible to suppose that, say, the axiom of infinity is pretheoretic or intuitive; but it is still very obviously part of FCUS.)

In any event, while FCNN contains the axioms and theorems of standard arithmetical theories, it is plausible to suppose that it also goes beyond those theories. For instance, one might hold that FCNN contains the sentence, 'The Gödel sentences of the standard axiomatic theories of arithmetic are true.' There are other sentences we might list here as well, but it is important to understand that we will not be able to get very precise about what exactly is contained in FCNN. For at least on the above way of conceiving of it and probably on any decent conception of it, FCNN is not a precisely defined set of sentences; on the contrary, there are numerous kinds of vagueness and imprecision here. One obvious issue is that 'uncontroversial' is a vague term. Another issue is that it is not clear when a sentence ought to count as being *about the natural numbers* in the relevant sense. For instance, according to my intuitions, sentences like '2 is not identical to the Mona Lisa' are about the natural numbers in the relevant sense, whereas sentences like '2 is the number of Martian moons' are not.[10] But others might feel differently about some such sentences. In any event, it seems likely to me that, in the end, there are some sentences for which there is simply no objective fact of the matter as to whether they are part of FCNN.

But this does not really matter. For whatever vaguenesses and imprecisions there are in FCNN, it is still (very obviously) strong and precise enough to pick out a unique mathematical structure up to isomorphism, and so platonists can claim that, in arithmetic, there is a unique intended structure up to isomorphism. (I suppose one might doubt the claim that FCNN picks out a unique structure up to isomorphism. Indeed, Putnam argued something like this in his (1980). I will not bother to respond

[8] There are various ways in which one might define fictionalistic correctness; below I define it as follows: A mathematical sentence is *fictionalistically correct* iff it would have been true if plenitudinous platonism had been true.

[9] Of course, none of this counts as a definition of 'accept'. I am not sure how exactly that term ought to be defined. One approach would be to take it as a primitive. Another approach would be to take it to be a natural kind term. I will not pursue this issue here.

[10] I think there is a view of aboutness that makes sense of these intuitions, but I do not have the space to develop this view here.

to this here because, in the present context, it does not really matter: if I became convinced that FCNN failed to pick out a unique structure up to isomorphism, I could just take the same line on arithmetic that I take below on set theory. I should say, however, that I think it is entirely obvious that FCNN *does* pick out a unique structure up to isomorphism, so that something must be wrong with Putnam's argument.[11,12])

In any event, whatever we say about arithmetic, in set theory, as we have already seen, it is not at all obvious that there is a unique intended structure up to isomorphism. There may be multiple parts of the mathematical realm such that (a) they are not isomorphic to one another, and (b) they all count as intended in connection with set theory, because they all fit perfectly with our full conception of the universe of sets, or FCUS. In particular, it may be that H1 and H2 (defined several paragraphs back) provide an instance of this, so that both of these hierarchies count as intended parts of the mathematical realm, and CH is true in some intended parts of the mathematical realm and false in others.

Given all of this, it seems to me that platonists should reject Better Platonism and endorse the following view instead:

> *IBP (short for "intention-based platonism"):* A pure mathematical sentence S is *true* iff it is true in *all* the parts of the mathematical realm that count as intended in the given branch of mathematics (and there is at least one such part of the mathematical realm); and S is *false* iff it is false in all such parts of the mathematical realm (or there is no such part of the mathematical realm[13]); and if S is true in some intended parts of the mathematical realm and false in others, then there is no fact of the matter whether it is true or false.

[11] Here is a quick argument for thinking that FCNN does pick out a unique structure up to isomorphism: while there are other structures in the vicinity—most notably, non-standard models of our first-order arithmetical theories—when someone points these structures out to us, our reaction is that they are clearly not what we had in mind, i.e., they are *unintended*. This alone suggests that these structures are inconsistent with our arithmetical intentions, or with FCNN (and since FCNN is not a first-order theory, there is no good reason to think that it has unintended (non-isomorphic) models). Now, someone like Putnam might raise an epistemological worry about how we humans could manage to zero our minds in here on a unique structure (up to isomorphism), but if what I argued earlier in this paper is correct, then plenitudinous platonists have a bead on how we are able to do this. Moreover, even if it were a complete mystery *how* we manage to do this, it is still obvious that we *do* manage to do it. In particular, it is obvious that nonstandard models of arithmetic are at odds with our arithmetical intentions. We know this first-hand–by simply noticing our intuitive reactions to non-standard models.

[12] It's worth noting that even if FCNN is *inconsistent*, it almost certainly still picks out a unique structure up to isomorphism. Indeed, it almost certainly still picks out the *right* structure, i.e., the one we think we have in mind in arithmetic. It is almost inconceivable that our natural-number thoughts are so badly inconsistent that it is not the case that we have in mind the structure that we think we have in mind. But if (against all appearances) this were indeed the case, it would not be a problem for platonists. It would be a problem for the mathematical community.

[13] We actually do not need this parenthetical remark, because if there are no such parts of the mathematical realm, then it will be true (vacuously) that S is false in all such parts of the mathematical realm. Also, one might want to say (à la Strawson) that if there is no such part of the mathematical realm, then S is neither true nor false, because it has a false presupposition. I prefer the view that in such cases, S is false, but nothing important turns on this.

Let us call platonists who endorse IBP (and also FBP) *IBP-platonists*. This view entails that there might be bivalence failures in mathematics, and so there are obvious worries one might have here, e.g., about the use of classical logic in mathematics. I will presently argue, however, that (a) IBP-platonism is perfectly consistent with the use of classical logic in mathematics and (b) the fact that IBP-platonism allows for the possibility of bivalence failures does not give rise to any good reason to reject the view, and indeed, it actually gives rise to a powerful argument in *favor* of the view.

The first point I want to make here is that while IBP-platonism allows for the possibility of bivalence failures in mathematics, it does not give rise to *wide-spread* bivalence failures. To begin with, there will not be any bivalence failures in arithmetic, according to this view. This is because (a) as we have seen, our full conception of the natural numbers—FCNN—picks out a unique structure up to isomorphism; and so (b) there is a unique intended structure for arithmetic (again, up to isomorphism); and it follows from this that (c) IBP-platonism does not allow any bivalence failures in arithmetic. Moreover, even when we move to set theory, IBP-platonism does not give rise to *rampant* bivalence failures. To begin with, I think it is pretty easy to argue that all of the standard axioms of set theory—including the axiom of choice—are inherent in FCUS, i.e., our full conception of the universe of sets, and so it seems that everything that follows from ZFC will be true in all intended structures and, hence, according to IBP-platonism, true. Likewise, everything that is inconsistent with ZFC will come out false on this view. In addition, IBP-platonism entails that set-theoretic sentences that are undecidable in ZFC can still be true. For instance, if it turns out that, unbeknownst to us, CH is built into FCUS, or if CH follows from some new axiom candidate that is built into FCUS, then CH is true in all intended parts of the mathematical realm, and so, according to IBP-platonism, it is true.

However, IBP-platonism also entails that it *might* be that there is no fact of the matter whether some undecidable sentences are correct. For instance, if CH and $\sim$CH are both fully consistent with FCUS, so that they are both true in some intended parts of the mathematical realm, then according to IBP-platonism, there is no fact of the matter as to whether CH is true. Now, it might seem that this is a problem for IBP-platonism, but I want to argue that, on the contrary, it actually gives rise to a powerful argument in its favor. For I think it can be argued that the following claim is true:

> In cases where IBP-platonism entails that there is no fact of the matter as to whether some mathematical sentence is true or false (or correct or incorrect), there *really is not* any fact of the matter as to whether the sentence in question is correct or incorrect, and so IBP-platonism dovetails here with the relevant facts.

Let me argue this point by focusing on the case of CH. If CH is neither true nor false according to IBP-platonism, then here is what we know: there are two hierarchies, or purported hierarchies—call them H1 and H2—such that CH is true in H1, $\sim$CH is true in H2, and H1 and H2 both count as intended hierarchies for set theory. Given this, how in the world could CH be correct or incorrect? Suppose you favored CH. How could you possibly mount a cogent argument against ~CH? If you found an axiom candidate A such that ZF + A entailed CH, then we know for certain that A would be wildly controversial; in particular, we know that ~A would be true in some intended structures. Now, of course, mathematicians might come to embrace A for some reason, but given what we are assuming about this scenario, it would not be plausible to take this acceptance of A as involving a discovery of an antecedently existing mathematical fact. It would rather involve some sort of change in the subject matter, or a decision to focus on a certain sort of theory, or a certain sort of structure, probably for some aesthetic or pragmatic reason. (This might lead to FCUS evolving so that, in the future, A was true, according to IBP-platonism. But it would not make it the case that A had been true all along.)

As far as I can see, there is only one way to avoid the conclusion that in the above H1-H2 scenario, there is no objective fact of the matter as to whether CH is correct. One would have to adopt a sparse platonist view according to which the CH question is settled by brute, arbitrary existence facts. Platonists might try to say that on their view, the solution to the CH problem, in the above H1–H2 scenario, is decided by what the actual set-theoretic universe is like—i.e., by whether CH or ~CH is true in *that* universe. But what does 'the actual set-theoretic universe' refer to? Presumably the *intended* structure for set theory—i.e., the structure we are talking about when we engage in 'set' talk. But in the above scenario, H1 and H2 both count as intended. So as long as H1 and H2 both exist, the CH question cannot be settled by looking at the nature of "the set-theoretic universe." Thus, it seems to me that there is only one way to obtain the result that in this scenario, there is an objectively correct answer to the CH question that is determined by facts about actually existing mathematical objects; one would have to say something like this:

> *Sparse Set-Theoretic Platonism (SSTP)*: It is not the case that H1-type hierarchies and H2-type hierarchies both exist. Hierarchies of one of these two kinds exist in the mathematical realm, but not both. If there exist H1-type hierarchies (and not H2-type hierarchies), then CH is true and ~CH is false; and if there exist H2-type hierarchies (and not H1-type hierarchies), then ~CH is true and CH is false.

But we are assuming here that platonists have good reason for rejecting sparse versions of platonism like this and endorsing FBP. One reason for this, as I have already made clear, is that FBP can be given an acceptable epistemology and sparse platonism cannot. I explained above what an FBP-ist epistemology would look like. I did not say anything there about why sparse platonism *cannot* be given an acceptable epistemology, but we can now see how one might construct an argument for this claim, because sparse platonist views like SSTP seem to make mathematical discovery impossible. Given that there is no contradiction in the idea of an H1-type hierarchy or an H2-type hierarchy, what grounds could we possibly have for believing in H1-type hierarchies but not H2-type hierarchies, or *vice versa*? After all, these are abstract objects we are talking about. What rational reason could we possibly have for believing in the existence of one mathematical structure but not another? The answer, it seems, is none.

SSTP is not just epistemologically unacceptable. It is also metaphysically bizarre. It seems to make mathematical truth completely arbitrary. Believing in H1-type hierarchies but not H2-type hierarchies is tantamount to believing in the platonic form *red* but not the platonic form *blue*. It is an arbitrary and bizarre view of the platonic realm.

Given the untenability of sparse platonist views like SSTP, it seems to me that there is no plausible view that delivers a guarantee that we will always have bivalence in mathematics. It really is true that there might be bivalence failures in connection with some undecidable sentences, and in particular, in the above scenario (i.e., the one in which H1 and H2 both count as intended set-theoretic hierarchies), there really is no fact of the matter as to whether CH is correct. Thus, the fact that IBP-platonism leads to these results is not a problem. Indeed, this counts in its favor because it is getting things right. (It is also worth noting that there is nothing special here about mathematics. In general, imprecision in our thought and talk can lead to bivalence failures, and that is all that is going on here.)[14]

This gives us one reason for favoring IBP-platonism over views that do not allow for the possibility of bivalence failures. Here is a second argument: IBP-platonism is *non-revisionistic* in the sense that it does not settle questions that are best settled by mathematicians. It is a controversial question whether there is an objective fact of the matter about the CH question, and it seems to me that this question should be settled by mathematicians, not philosophers. IBP-platonism allows mathematicians to settle this question, because it entails that this is an open mathematical question. But most philosophies of mathematics are more dictatorial here; e.g., most traditional versions of realism entail that there is a fact of the matter about CH, and most traditional versions of anti-realism entail that there is not.

Finally, it is important to note that IBP-platonism is perfectly consistent with the use in mathematics of classical logic, in particular, the law of excluded middle. According to IBP-platonism, we can safely use the law of excluded middle in proofs, because all (purely mathematical) instances of that law are true, because they are all true in all intended structures. Now, of course, according to IBP-platonism, this is a sort of half truth, because that view entails that there might be some mathematical sentences that are neither true nor false. But this just does not matter, because IBP-platonism entails that mathematicians can use the law of excluded middle without being led astray anyway.[15]

[14] Tony Martin once told me that on his view, there *is* an objectively correct answer to the CH question because we can just ask whether CH is true of the part of the mathematical realm that contains *all* the sets, from all the different hierarchies. But if our term 'set' is imprecise, then there is no fact of the matter as to which part of the mathematical realm is the part that contains all and only sets. Now, Martin might doubt that 'set' is imprecise, and he might be right, but that is irrelevant here, because I am just saying what would follow if it *were* imprecise.

[15] You might object here as follows: "Since *some* mathematicians think that it is not legitimate to use the law of excluded middle in mathematics, it seems that *some* mathematicians would see IBP-platonism as revisionistic." I cannot fully respond to this here, but it seems to me that the arguments that some people give against the legitimacy of the law of excluded middle are not mathematical arguments at all; they are philosophical arguments. So it seems to me that the anti-excluded-middle view is *itself* a kind of revisionism—it advocates a revision to classical mathematics based on philosophical arguments (and I would add here that, on my view, the philosophical arguments in question are *bad* arguments).

IBP-Platonism and Mathematical Pluralism

Given that platonists should endorse IBP-platonism, it should be obvious how their view entails that a kind of pluralism is true. It does not entail the mad-dog, foaming-at-the-mouth pluralism that Silly Platonism entails—i.e., that every consistent mathematical theory is true. But it does entail that a more restricted kind of pluralism is true.

Consider ZF + CH and ZF + ∼CH. IBP-platonism does not entail that both of these theories are true. But it does entail that *in certain situations* they would both be true. Here are two such situations:

1. *The different-concepts situation*: Martians have a slightly different concept of *set* than we do. Their concept of *set* picks out hierarchies in which ZF + CH is true; our concept (let us assume) picks out hierarchies in which ZF + ∼CH is true. So in their mouths ZF + CH is true, and in our mouths ZF + ∼CH is true. We get mathematical pluralism because we have two cultures with different concepts and, hence, different intentions.
2. *The different-axiom-systems situation*: Two mathematicians, M1 and M2, are "playing around with set-theoretic axiom systems." M1 is studying the system ZF + CH; she is explicitly *not* trying to study "the universe of sets," or to capture our intuitive concept of a set; all she wants to do is explore hierarchies that are characterized by ZF + CH. Likewise, M2 is studying the system ZF + ∼CH; he is explicitly *not* trying to study "the universe of sets," or to capture our intuitive concept of a set; all he wants to do is explore hierarchies that are characterized by ZF + ∼CH. So for M1, the intended structures are the ones that are characterized by ZF + CH, and for M2, the intended structures are the ones that are characterized by ZF + ∼CH. So in M1's mouth ZF + CH is true, and in M2's mouth ZF + ∼CH is true. We get mathematical pluralism because we have two people playing around with different axioms systems and, hence, two people with different intentions.

So we have a kind of pluralism here, and we have a kind of relativism. All we need in order to get this result is two different people (or groups of people) with different intentions.

Does this view entail a contradiction? No. Because the two different people (or groups of people) are talking about different objects. Suppose I am talking about Bill Clinton and I say, "Clinton ran for president in 1992"; and suppose you are talking about Hilary Clinton and you say, "Clinton didn't run for president in 1992." Both of our utterances are true. Does this entail a contradiction? No. Because our sentences are about different objects. Likewise for the two scenarios above. M1 and M2 are talking about different hierarchies, and so are the humans and the Martians.

There are two different worries that one might raise at this point. We can put them like this:

Worry 1: The kind of pluralism (and relativism) that we get from IBP-platonism is not a genuine, robust kind of pluralism (or relativism).
Worry 2: IBP-platonism is not a genuine, robust kind of platonism. It gives us a platonistic *ontology*, but it does not give us a robust kind of *objectivity*.

Let me address worry 2 first. It seems to me that the claim that IBP-platonism does not give us a robust kind of objectivity is just false. It gives us the same sort of objectivity that we get anywhere else. Consider, e.g.,

(M) Mars is round.

In our language, this is true because (a) we use it to say that a certain object (Mars) has a certain property (roundness), and (b) that object does have that property. Of course, Martians could use it to say something else—e.g., that Mars is flat—and if they did then in their mouths (M) would be false. But this does not mean that (M) is not objectively true in our language. And likewise for mathematical sentences like CH and $\sim$CH: if Martians use CH to say something objectively true about one part of the mathematical realm, it does not follow that we cannot use $\sim$CH to say something objectively true about another part of the mathematical realm.

This response to worry 2 might make worry 1 seem more pressing. For despite the above—despite the fact that in Martianese (M) could say something that is false, e.g., that Mars is flat—we would not say that we are *relativists* about (M). My response to this is that the mathematical case is different. Given that IBP-platonism is a *plenitudinous* version of platonism, we get the following result: *which (consistent) mathematical sentences are true is wholly determined by our intentions*. This is because (a) mathematical truth comes down to truth in intended structures, and (b) which mathematical structures count as intended is wholly determined by our intentions. Nothing analogous to this is true of sentences like (M)—i.e., sentences about the physical world. Which (consistent) sentences about the physical world are true is determined not just by our intentions but by external facts. But this is not the case on the IBP-platonist view; facts about the mathematical realm do not do *anything* to determine which (consistent) mathematical sentences are true.

This gives us a kind of mathematical pluralism (and mathematical relativism) that does not seem to be true in connection with sentences about the physical world. It is hard to articulate exactly what this pluralism/relativism consists in, but the basic idea should be clear: Whatever they say about the mathematical realm, Martians can construct a true mathematical theory by simply constructing a consistent axiom system, and intending to be talking about whatever objects satisfy those axioms. And, of course, we can do the same. And this seems (to me anyway) to give us a kind of pluralism/relativism that is not true of ordinary talk about the physical world.

Pluralism and Anti-platonism

I now want to argue that even if we abandon platonism, we should still endorse pluralism—indeed, the exact same kind of pluralism/relativism that I have argued we will get if we endorse platonism. If this is right, then we have good reason to endorse this sort of pluralism/relativism regardless of what we say about the ontology of mathematics.

I think there are very good (empirical) arguments for the platonistic interpretation of mathematical sentences like '3 is prime'; in other words, I think there are good arguments for the claim that these sentences are supposed to be about abstract objects. I will not give the argument for this claim here; I will just assume that it's right. If this is true, then if we reject platonism—if we maintain that there are no such things as abstract objects—then we have to endorse one of the following three views:

> *Mathematical Fictionalism (aka Error Theory)*: (a) Our mathematical theories do purport to be about abstract objects, as platonists claim (e.g., '3 is prime' should be interpreted as purporting to make a claim about the number 3); but (b) there are no such things as abstract objects; and so (c) our mathematical theories are not literally true. Thus, on this view, just as *Alice in Wonderland* is not true because (among other reasons) there are no such things as talking rabbits, hookah-smoking caterpillars, and so on, so too our mathematical theories are not true because there are no such things as numbers and sets and so on.[16]
>
> *Meinongianism*: (a) Our mathematical theories do purport to be about abstract objects, as platonists claim (e.g., '3 is prime' should be interpreted as purporting to make a claim about the number 3); but (b) abstract objects do not exist (although they *are*, in some sense); and so (c) our mathematical theories are about (and in fact, are true descriptions of) objects that do not exist.[17]
>
> *Azzouni-style objectless-truth-ism*: (a) Our mathematical theories do purport to be about abstract objects, as platonists claim (e.g., '3 is prime' should be interpreted as purporting to make a claim about the number 3); and (b) there are no such things as abstract objects; but nevertheless, (c) our mathematical theories are still true.[18]

I think that advocates of all three of these views are committed to endorsing a pluralism/relativism of essentially the same kind that IBP-platonists are committed to. I will focus mainly on the case of fictionalists, and I will start with them. There are a few different challenges to fictionalism, but one of the main ones is that fictionalists need to find some plausible way to account for the fact that mathematics is an objective discipline. If fictionalism is right, then sentences like '3 is prime' and '4 is prime' are both false. But there is obviously some important difference

[16] This view was introduced by Field (1980) and developed further by, e.g., me (1998) and Leng (2010). Similar views are endorsed by Melia (2000), Rosen (2001), Yablo (2002), and Bueno (2009).

[17] See, e.g., Meinong (1904), Routley (1980), Parsons (1980), and Priest (2005).

[18] See Azzouni (2004).

between them. We cannot just say that they are both mistakes. We cannot throw the entire discipline of mathematics into the garbage. So fictionalists need to find some way of accounting for the difference between sentences like '3 is prime' and sentences like '4 is prime.' To put the point differently, it seems that even if '3 is prime' is not strictly true, there is obviously some important sense in which it is *right*, or *correct*, and fictionalists need to provide some account of this sort of correctness.

Field (1980) responded to this worry by claiming that the difference between '3 is prime' and '4 is prime' is analogous to the difference between 'Oliver Twist grew up in London' and 'Oliver Twist grew up in L.A.' In other words, the difference is that '3 is prime' is part of a certain well-known mathematical story, whereas '4 is prime' is not. Field expressed this idea by saying that while neither '3 is prime' nor '4 is prime' is true *simpliciter*, there is another truth predicate (or pseudo-truth predicate, as the case may be)—viz., 'is true in the story of mathematics'—that applies to '3 is prime' but not to '4 is prime.' And this, fictionalists might say, is why '3 is prime' is correct—or *fictionalistically correct*—and '4 is prime' is not.

This, I think, is a good start, but fictionalists need to say more. In particular, they need to say what the so-called "story of mathematics" consists in. Field's view [see, e.g., his (1998)] is that it consists essentially in the formal axiom systems that are currently accepted in the various branches of mathematics. But this view is problematic. One might object to it as follows:

> Field's view enables fictionalists to account for the correctness of sentences like '3 is prime', but there is more than this to the objectivity of mathematics. In particular, it seems that objective mathematical correctness can outstrip currently accepted axioms. For instance, it could turn out that mathematicians are going to discover an objectively correct answer to the question of whether CH is true or false. Suppose, for instance, that (i) some mathematician M found a new set-theoretic axiom candidate A that was accepted by mathematicians as an intuitively obvious claim about sets, and (ii) M proved CH from ZF + A. Then mathematicians would say that M had *proven* CH, that she had *discovered* the answer to the CH question, and so on. Indeed, given what we are assuming about A—that it's an intuitively obvious claim about sets—it would not even occur to mathematicians to say anything else. And it is hard to believe they would be mistaken about this. The right thing to say would be that CH had been correct all along and that M came along and discovered this. But Field can not say this. Given that CH and $\sim$ CH are both consistent with currently accepted set theories, he is committed to saying that neither is true in the story of mathematics and, hence, that there is (at present) no objectively correct answer to the CH question, so that no one could discover the answer to that question.

In order to respond to this objection, fictionalists need a different theory of what the story of mathematics consists in. The fictionalist view I want to develop is based on the following claim:

> The story of mathematics consists in the claim that there actually exist abstract mathematical objects of the kinds that platonists have in mind—i.e., the kinds that our mathematical theories are about, or at least purport to be about.

This view gives rise to a corresponding view of fictionalistic mathematical correctness, which can be put like this:

> A pure mathematical sentence is correct, or *fictionalistically correct*, iff it is true in the story of mathematics, as defined in the above way; or, equivalently, iff it would have been true if there would have actually existed abstract mathematical objects of the kinds that platonists have in mind, i.e., the kinds that our mathematical theories purport to be about.

The nice thing about this view is that it enables fictionalists to essentially steal whatever platonists say about mathematical truth. Since fictionalists say that the correct sentences are just the sentences that would have been true if platonism had been true, they should be able to endorse a theory of mathematical correctness that mirrors the platonist theory of mathematical truth. This is very convenient for us because we have already seen what platonists should say about mathematical truth—they should endorse IBP (and FBP). Thus, fictionalists should follow an analogous strategy. First, they should say that the story of mathematics consists in the claim that FBP is true—i.e., that there is a *plenitudinous* mathematical realm—and so they should say that a mathematical sentence is *correct* (or true in the story of mathematics) iff it would have been true if FBP had been true. And second, they should endorse the following analogue of IBP:

> (IBF) A pure mathematical sentence S is *correct* iff, in the story of mathematics, S is true in all the parts of the mathematical realm that count as intended in the given branch of mathematics; and S is *incorrect* iff, in the story of mathematics, S is false in all intended parts of the mathematical realm; and if, in the story of mathematics, S is true in some intended parts of the mathematical realm and false in others, then there is no fact of the matter as to whether S is correct or incorrect.

Let us say that fictionalists who endorse this view are *IBF-fictionalists*. I want to make two points about this view. First, it avoids the above problem with Field's view. In the scenario in which M proves CH from ZF + A, we can reason as follows:

> The fact that A strikes mathematicians as an intuitively obvious claim about sets suggests that it's part of FCUS—i.e., part of our full conception of the universe of sets. But this suggests that it is true in all intended structures (or more precisely, that *in the story of mathematics*, it is true in all intended structures). Thus, since ZF is also presumably true in all intended structures, we can say that anything that follows from ZF + A is true in all intended structures. Thus, CH is true in all intended structures. Or again, to be more precise, we can say that *in the story of mathematics*, CH is true in all intended structures. And so it follows that CH is fictionalistically correct. And note that on the IBF-fictionalist view, CH did not *become* correct when M proved it. M

> *discovered* that CH was correct all along. We just had not noticed this because we had not noticed that A was correct and that ZF + A entails CH.

The second point I want to make about IBF-fictionalism is that it entails the following claim: *which (consistent) mathematical sentences are correct is wholly determined by our intentions*. This is exactly parallel to the fact that IBP-platonism entails that which (consistent) mathematical sentences are *true* is determined by our intentions. But this means that we are going to get the exact same sort of pluralism/relativism that we got with platonism. We do not get the crazy result that *all* mathematical sentences are correct. But we do get the result that there could be certain situations in which, e.g., CH and ∼CH are both correct; more specifically, we get the result that in the above scenario in which humans and Martians have slightly different concepts of set (and, hence, slightly different intentions), CH is correct in the mouths of Martians, and ∼ CH is correct in the mouths of humans. Everything is exactly the same, and so we get the exact same kind of pluralism/relativism.

Let us move on now to Meinongianism. It seems to me that just about everything we said about platonists applies equally to them. First, it seems obvious that Meinongians should say that there is a *plenitude* of mathematical objects that do not exist. Thus, it seems that on this view, all consistent purely mathematical theories (and perhaps some inconsistent ones as well) accurately characterize collections of abstract objects (that do not exist). Thus, since Meinongians presumably will not want to say that all such theories are true—since they will not want to endorse what we might call *Silly Meinongianism*—they will have to provide a theory of mathematical truth that tells us *which* of the various consistent theories are true. And I think that the considerations that pushed platonists to endorse IBP-platonism will push Meinongians to endorse an analogous view—i.e., I think they will end up saying that a mathematical theory is true iff it is true in all intended structures. And this will lead to the same result that platonists are led to—*that which (consistent) theories are true is wholly determined by our intentions*. And this, in turn, will lead Meinongians to the same sort of pluralism/relativism that IBP-platonists and IBF-fictionalists are led to.

Finally, I think an exactly parallel sort of argument can be used to show that Azzouni's view leads to the same sort of pluralism/relativism, but I will not argue the point here.

Conclusion

In this paper, I have articulated a certain kind of mathematical pluralism/relativism, and I have argued that platonists and anti-platonists should both endorse this view. Thus, if my arguments are cogent, then we have good reason to think that this sort of pluralism/relativism is true.

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