Algebra Comprehensive Exam
Sample Test

Note: This sample algebra comprehensive exam is meant to give a sense of the format for the new algebra comprehensive exam syllabus that will be in effect for students in the masters degree program beginning with the 2020–2021 academic year.

Answer five (5) questions only. You must answer *at least one* from each of section: (I) Linear algebra, (II) Group theory, and (III) Synthesis: linear algebra and group theory. Indicate CLEARLY which problems you want us to grade; otherwise, we will select the first problem from each section, and then the first two additional problems answered after that. Be sure to show enough work that your answers are adequately supported. Tip: When a question has multiple parts, the later parts often (but not always) make use of the earlier parts.

Notation: Unless otherwise stated, $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_n, \mathbb{C}$, and $\mathbb{R}$ denote the sets of rational numbers, integers, integers modulo n, complex numbers, and real numbers respectively, regarded as groups in the usual way.

Part I. Linear algebra

(1) Let V be a vector space over a field F. Let W be a subspace of V. Fix an element $v_0 \in V$. Define the set

$$W_0 = \{v_0 + w \mid w \in W\}$$

Prove that W_0 is a subspace of V if and only if $v_0 \in W$.

(2) Let V and W be finite-dimensional vector spaces over a field F. Let $T : V \to W$ be a linear transformation that is one-to-one and onto. Prove: If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then $\beta' = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

(3) Let V be a vector space and F be a field. Let v, w, z be vectors in V. Suppose that $S = \{v, w\}$ is a linearly independent set. Prove that $\{v, w, z\}$ is linearly dependent if and only if z is in the span of S.

Part II. Group theory

(1) Let G be a group and p be a prime number.

(a) Prove or disprove: If the order of G is p^2, then G is abelian.
(b) Prove or disprove: If the order of G is p^3, then G is abelian.

(2) Let G be a group with identity element e. Let $x \in G$ be an element of finite order n. Prove that $x^m = e$ if and only if n divides m.

(3) Let $\mathbb{R}^\times$ be the set of nonzero real numbers.

(a) Prove that $\mathbb{R}^\times$ is an abelian group under multiplication.
(b) Let $H = \{1, -1\}$. Let $\mathbb{R}^+$ be the set of positive real numbers. Prove that H and $\mathbb{R}^+$ are normal subgroups of $\mathbb{R}^\times$.

(c) Use the First Isomorphism Theorem to prove that $\mathbb{R}^\times/H$ is isomorphic to $\mathbb{R}^+$.

Part III. Synthesis: Linear algebra and group theory

(1) Let V be a vector space. Let G be the set of all bijective linear transformations from V to V. Prove that G is a group under function composition.

(2) Recall that $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ matrices under matrix multiplication. Let

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

Prove that H is a subgroup of $GL(3, \mathbb{R})$.

(3) For any positive integer n, recall that $O(n)$ is the group of $n \times n$ orthogonal matrices.

(a) Prove that the map $A \cdot v \mapsto Av$ defines a group action of $O(n)$ on $\mathbb{R}^n$. Here we regard elements of $\mathbb{R}^n$ as column vectors, and Av denotes the product of the matrix A and the vector v under ordinary matrix multiplication.

(b) For $n = 2$, describe the orbit of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under this action. Draw a picture to illustrate your answer.