No-hole 2-distant colorings for unit interval graphs

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December 23, 1999

Abstract

Given a graph, a no-hole 2-distant coloring (also called N-coloring) is a function f that assigns to each vertex a non-negative integer (color) such that the separation of the colors of any pair of adjacent vertices must be at least 2, and all the colors used by f form a consecutive set (the no-hole assumption). The minimum consecutive N-span of G, $\mathrm{csp}_1(G)$, is the minimum difference of the largest and the smallest colors used in an N-coloring of G, if there exists such a coloring; otherwise, define $\mathrm{csp}_1(G) = \infty$. Here we investigate the exact values of $\mathrm{csp}_1(G)$ for unit interval graphs (also known as 1-unit sphere graphs). Earlier results by Roberts [18] indicate that if G is a unit interval graph on n vertices, then $\mathrm{csp}_1(G)$ is either $2\chi(G) - 1$ or $2\chi(G) - 2$, if $n > 2\chi(G) - 1$; $\mathrm{csp}_1(G) = \infty$, if $n < 2\chi(G) - 1$, where $\chi(G)$ denotes the chromatic number. We show that in the former case (when $n > 2\chi(G) - 1$), both values of $\mathrm{csp}_1(G)$ are attained, and give several families of unit interval graphs such that $\mathrm{csp}_1(G) = 2\chi(G) - 2$. In addition, the exact values of $\mathrm{csp}_1(G)$ are completely determined for unit interval graphs with $\chi(G) = 3$.

1 Introduction

The no-hole 2-distant coloring is originated from T-coloring, a channel assignment problem introduced by Hale [7]. Suppose several transmitters or stations, and a forbidden set T (called T-set) of non-negative integers

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(with $0 \in T$) are given. We need to assign to each transmitter or station a non-negative integral channel under the constraint that if two transmitters interfere, then the difference of their channels does not fall within the T-set. Two transmitters may interfere due to various reasons such as geographic proximity and meteorological factors. To formulate this problem, we construct a graph G such that each vertex represents one transmitter, and two vertices are adjacent if their corresponding transmitters interfere.

Thus, we have the following definition. Given a T-set and a graph G, a *T-coloring* of G is a function $f : V(G) \to Z^+ \cup \{0\}$ such that

$$|f(x) - f(y)| \notin T \quad \text{if} \quad xy \in E(G).$$

A *no-hole T-coloring* of G is a T-coloring f such that $f(V)$ is a consecutive set.

The *span* of a T-coloring f is the difference of the largest and the smallest colors used in $f(V)$. The *T-span* of a graph G, $\mathrm{sp}_T(G)$, is the minimum span among all possible T-colorings of G. The variable T-span for different graphs and different T-sets has been studied extensively by several authors (see [2, 3, 4, 6, 11, 12, 13, 15, 16, 20]).

It is known [3] that for any given T-set and graph G, a T-coloring always exists. However, a no-hole T-coloring does not have this property. For instance, take $T = \{0, 1\}$ and $G = K_2$. Hence, we define the *consecutive T-span* of a graph G, denoted by $\mathrm{csp}_T(G)$, by the minimum span of a no-hole T-coloring if there exists such a coloring; and define $\mathrm{csp}_T(G) = \infty$ otherwise.

For the case that $T = \{0, 1\}$ and $T = \{0, 1, 2, \ldots, r\}$, a no-hole T-coloring is also called an N-coloring (in [18]) and an N_r-coloring (in [19]), respectively. That is, an N_r-coloring of a graph G is a function $f : V(G) \to Z^+ \cup \{0\}$ such that $f(V)$ is consecutive and it satisfies the condition

$$|f(x) - f(y)| \geq r + 1, \quad \text{if} \quad xy \in E(G).$$

Roberts [18] and Sakai and Wang [19] studied the N-coloring and the N_r-coloring, respectively. Among the findings in [18, 19] are the results about the existence of an N-coloring and an N_r-coloring, respectively, for special graphs such as paths, cycles, bipartite graphs and 1-unit sphere graphs. Moreover, if it is the case that such a coloring exists, the authors also gave upper and lower bounds of the span.

The exact values of $\mathrm{csp}_T(G)$ for some families of graphs and T-sets were studied by Liu and Yeh [14] in which the authors proved: If T is r-initial (i.e. $T = \{0, 1, 2, \ldots, r\} \cup A$, where A contains no multiple of $(r + 1)$) or $T = \{0, a, a + 1, a + 2, \cdots, b\}$, then for any large n, there exists a graph G on n vertices such that $\mathrm{csp}_T(G) = n - 1$. The exact values of $\mathrm{csp}_r(G)$ for bipartite graphs were investigated by Chang, Juan and Liu [1]. In [1], the authors determined the values of $\text{csp}_r(G)$ for all bipartite graphs with at least $r-2$ isolated vertices, and completely determined $\text{csp}_2(G)$ for bipartite graphs.

A graph $G = (V, E)$ is a *k-unit sphere graph* if there is a function g from $V(G)$ into the Euclidean k-space R^k such that for all $x \neq y$ in V, $xy \in E$ if and only if $d(g(x)-g(y)) \leq 1$, where d denotes the Euclidean distance between two points in R^k. The 1-unit sphere graphs are also known as *unit interval graphs* or *indifference graphs* in the literature (see [5]).

If $T = \{0, 1, 2, \ldots, r\}$, denote $\text{csp}_T(G)$ by $\text{csp}_r(G)$. In this article, we focus on the exact values of $\text{csp}_1(G)$ for unit interval graphs. In Section 2, we cite some known results in T-colorings and no-hole T-colorings that will be used later in our proofs. Section 3 is focused on the computation of the exact values of $\text{csp}_1(G)$ for unit interval graphs G. In particular, $\text{csp}_1(G)$ is obtained for some families of unit interval graphs, and $\text{csp}_1(G)$ is completely determined for unit interval graphs with $\chi(G) = 3$, where $\chi(G)$ denotes the chromatic number of G.

2 Preliminaries

It is well-known [3, 10] that if T is r-initial, then the following holds:

$$\text{sp}_T(G) = (\chi(G) - 1)(r + 1) \text{ for all graphs } G. \qquad (*)$$

By the definition of a no-hole T-coloring, if $\text{csp}_T(G)$ is finite, a trivial upper bound for $\text{csp}_T(G)$ is $n-1$, where $n = |V(G)|$. Since any no-hole T-coloring is also a T-coloring, by $(*)$, we have:

Proposition 1 *For any positive integer r and any graph G on n vertices. If $\text{csp}_r(G) < \infty$, then $(\chi(G) - 1)(r + 1) \leq \text{csp}_r(G) \leq n - 1$.*

It is well-known that unit interval graphs are perfect (see [5]), hence for any unit interval graphs G, $\chi(G) = \omega(G)$, where $\omega(G)$ is the size of a maximum clique in G. Another well-known result that will be used in this article is due to Roberts [17]: A graph $G = (V, E)$ is a unit interval graph if and only if it has a *compatible vertex ordering*, i.e. an ordering $v_1, v_2, \ldots, v_n$ of vertices of G so that if $i < j < k$ and $v_i v_k \in E$, then $v_i v_j, v_j v_k \in E$.

Using the compatible vertex ordering of a unit interval graph, Roberts [18] proved implicitly, without mentioning the variable $\text{csp}_1(G)$, the following:

Theorem 2 ([18]) *If G is a unit interval graph on n vertices, then*

$$\text{csp}_1(G) \begin{cases} \leq 2\chi(G) - 1, & \text{if } n > 2\chi(G) - 1; \\ = \infty, & \text{if } n < 2\chi(G) - 1. \end{cases}$$

The theorem above was extended by Sakai and Wang [19] who showed the following:

Theorem 3 **([19])** *If G is a unit interval graph on n vertices, then*

$$\mathrm{csp}_r(G)\begin{cases} \leq (r+1)\chi(G)-1, & \text{if } n \geq (r+1)\chi(G); \\ = \infty, & \text{if } n \leq (r+1)(\chi(G)-1). \end{cases}$$

Figure 1 shows an example of Theorem 3.

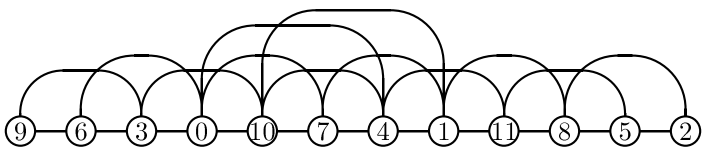

Figure 1: A unit interval graph with $\chi(G) = 4$ and $\mathrm{csp}_2(G) = 11$.

Although from the theorem above the problem of determining the existence of an N_r-coloring is not completely settled for general values of r, for $r = 1$, referring to Theorem 2, Sakai and Wang [19] completed the answer by confirming the case $n = 2\chi(G) - 1$:

Theorem 4 **([19])** *If G is a unit interval graph on $n = 2\chi(G)-1$ vertices, then*

$$\mathrm{csp}_1(G) = \begin{cases} 2\chi(G)-2, & \text{if } G \text{ has a unique maximum clique;} \\ \infty, & \text{otherwise.} \end{cases}$$

3 Main results

In this section, we investigate the exact values of $\mathrm{csp}_1(G)$ for unit interval graphs G. According to Theorems 2 and 4, we consider unit interval graphs with more than $2\chi(G) - 1$ vertices. By Proposition 1 and Theorem 2, the only possible values of $\mathrm{csp}_1(G)$ for such graphs are $2\chi(G)-2$ and $2\chi(G)-1$. We show both values are attainable, and give complete solutions of $\mathrm{csp}_1(G)$ for unit interval graphs with $\chi(G) = 3$.

Without loss of generality, all the graphs considered in this section are simple and connected. Throughout the section, unless indicated, we suppose $G = (V, E)$ is a unit interval graph with a compatible vertex ordering $P = v_1, v_2, \ldots, v_n$, where $n = |V(G)| > 2\chi(G) - 1$. The *distance* of two vertices v_i and v_j on P, denoted by $\mathrm{d}_P(u, v)$, is defined as $|i - j|$. And we let

$$A := \{v : \ v \ \text{ is in some maximum clique of } \ G\}; \ B := V(G) - A.$$

Theorem 5 *Suppose $G = (V, E)$ is a unit interval graph on n vertices, $n > 2\chi(G) - 1$. If $|B| < \chi(G) - 1$, then $\mathrm{csp}_1(G) = 2\chi(G) - 1$.*

Proof. Suppose to the contrary that $\mathrm{csp}_1(G) = 2\chi(G) - 2$, and let f be an N-coloring of G, $f : V(G) \to \{0, 1, 2, \ldots, 2\chi(G) - 2\}$. Since $\chi(G) = \omega(G)$, we have $|f(u) - f(v)| \geq 2$ for any maximum clique W and $u, v \in W$. This implies that $f(x) \in \{0, 2, 4, \ldots, 2\chi(G) - 2\}$ for all $x \in A$. Hence there must exist at least $\chi(G) - 1$ vertices in B that are labeled by $\{1, 3, \ldots, 2\chi(G) - 3\}$, contradicting the assumption $|B| < \chi(G) - 1$. Therefore, $\mathrm{csp}_1(G) = 2\chi(G) - 1$. □

Theorem 6 *Suppose $G = (V, E)$ is a unit interval graph on $n > 2\chi(G) - 1$ vertices and $P = v_1, v_2, \ldots, v_n$ is a compatible vertex ordering of G. If $\chi(G) = m \geq 3$ and there exists a subset $\{v_{s+1}, v_{s+2}, \ldots, v_{s+m-1}\} \subseteq B$ for some $0 \leq s \leq n - m + 1$, then $\mathrm{csp}_1(G) = 2m - 2$.*

Proof. It suffices to find an N-coloring for G with span $2m - 2$. We define a coloring f by first labeling the vertices $v_{s+1}, v_{s+2}, \ldots, v_{s+m-1} \in B$ by $f(v_{s+i}) = 2i - 1$, $1 \leq i \leq m - 1$, that is $f(B) = \{1, 3, 5, \ldots, 2m - 3\}$. Secondly, label the vertices preceding v_{s+1} (if there is any), backwards, by repeating the pattern of colors $\ll 2m - 2,\ 2m - 4,\ \ldots, 4, 2, 0 \gg$ (i.e., $f(v_s) = 2m - 2$, $f(v_{s-1}) = 2m - 4$, $f(v_{s-2}) = 2m - 6$, etc., until v_1 is colored). Finally, repeat the pattern of colors $\ll 0, 2, 4, \ldots, 2m-4, 2m-2 \gg$ to the remaining vertices (i.e., $f(v_{s+m}) = 0$, $f(v_{s+m+1}) = 2$, etc., until the last vertex v_n is colored). See Figure 2 as an example.

Because $n \geq 2m - 1$, the even colors $0, 2, \ldots, 2m - 2$ are all used by f. Combining this with the fact that $f(B) = \{1, 3, 5, \ldots, 2m - 3\}$, f is onto with span $2m - 2$. It is not hard to verify that f is indeed an N-coloring. We leave the details to the reader. □

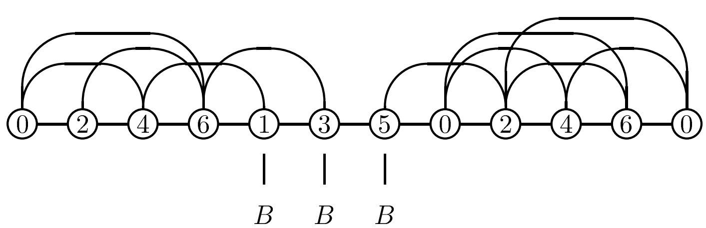

Figure 2: A unit interval graph with $\chi(G) = 4$ and $\mathrm{csp}_1(G) = 6$.

Theorem 7 *Suppose G is a unit interval graph on n vertices with $\chi(G) = m$ and $n > 2m-1$. If there exists a compatible vertex ordering $P = v_1, v_2, \ldots, v_n$ such that $A \subseteq \{v_i, v_{i+1}, \ldots, v_j\}$, where $i \geq 1$, $j-i+1 = km$ for some positive integer k, and $n \geq (k+1)m-1$, then $\mathrm{csp}_1(G) = 2m-2$.*

Proof. It suffices to find an N-coloring for G with span $2m-2$. Define the coloring function f by first labeling vertices $v_i, v_{i+1}, \ldots, v_j$ by using the pattern $\ll 0, 2, 4, \ldots, 2m-2 \gg$. (i.e. $f(v_i) = 0$, $f(v_{i+1}) = 2$, etc.) Then $f(v_j) = 2m-2$, since $j-i+1 = km$ for some positive integer k.

Next, label the vertices prior to v_i (if there is any) by the pattern $\ll 2m-3, 2m-5, \ldots, 5, 3, 1 \gg$, backwards, until the first vertex on P is labeled. Finally, label the vertices after v_j (if there is any) by the pattern $\ll 1, 3, 5, \ldots, 2m-5, 2m-3 \gg$ until the last vertex is labeled. By the assumptions that $m \geq 3$, $A \subseteq \{v_i, v_{i+1}, \ldots, v_j\}$, and $n \geq (k+1)m-1$, it is easy to verify that f is an N-coloring. $\square$

Corollary 8 *If G is a unit interval graph with a unique maximum clique, then*

$$\mathrm{csp}_1(G) = \begin{cases} 2\chi(G)-2, & \textit{if } n \geq 2\chi(G)-1; \\ \infty, & \textit{otherwise.} \end{cases}$$

Proof. The result follows from Theorems 2, 4, and Theorem 7. $\square$

Theorem 6 gives a result for the case that B contains a subset of *consecutive* $\chi(G)-1$ vertices on a compatible vertex ordering. In the next theorem, we prove that, under some conditions, the same result also holds when B has vertices that are scattered along the compatible vertex ordering. This result is also a generalization of Theorem 7.

Theorem 9 *Suppose G is a connected unit interval graph on n vertices, $\chi(G) = m \geq 3$, $n > 2m-1$, G has a compatible vertex ordering $P = v_1, v_2, \ldots, v_n$, and there exists $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{m-1}}\} \subseteq B$, where $1 \leq i_1 < i_2 < \ldots < i_{m-1} \leq n$. Then $\mathrm{csp}_1(G) = 2m-2$ if there exist $1 \leq a, b < m-1$ such that $p = (m-1-a-b)/2$ is a non-negative integer, and*

$$i_{j+1} = \begin{cases} i_j + k_j m + 1 \textit{ for some positive integer } k_j, & \textit{if } j \in C; \\ i_j + 1, & \textit{if } j \notin C, \end{cases}$$

where $C = \{1, \ldots, p, p+a, p+a+b, p+a+b+1, \ldots, m-2\}$ (if $p = 0$, then $C = \{a\}$).

Proof. It suffices to find an N-coloring of G with span $2m-2$. We define the coloring $f : V(G) \to \{0, 1, 2 \dots, 2m-2\}$ by:

$$f(v_{i_j+k}) = \begin{cases} 4j + 2b - 3, & \text{if } 1 \le j \le p,\ k = 0; \\ 2(j-p-a) - 3, & \text{if } p+1 \le j \le p+a,\ k = 0; \\ 2(j-p-a) - 1, & \text{if } p+a+1 \le j \le p+a+b,\ k = 0; \\ 4(j-p-a) - 2b - 1, & \text{if } p+a+b+1 \le j \le m-1,\ k = 0; \\ f(v_{i_j}) + 1 + 2k, & \text{if } j \in C \cup \{m-1\}, \text{and} \\ & 1 \le k \le i_{j+1} - i_j - 1 \quad (i_m = n+1). \end{cases}$$

then color the vertices preceding v_{i_1} (if there is any) by repeating the pattern $\ll 2b-2, 2b-4, \dots, 2b \gg$, backwards, until the first vertex is colored. All the colors by f above are taken under modular $2m$. See Figure 3 for an example.

We call the set of vertices $\{v_{i_j+1}, v_{i_j+2}, \dots, v_{i_{j+1}-1}\}$ *block* B_j for each $j \in C$, the vertices preceding v_{i_1} (if there is any) block B_0, and the vertices after $v_{i_{m-1}}$ (if there is any) block B_{m-1}. Note that $V(G) = \{v_{i_1}, v_{i_2}, \dots, v_{i_{m-1}}\} \cup B_0 \cup_{j \in C} B_{i_j} \cup B_{m-1}$.

The last case in the function f defined above gives labels for vertices in those B blocks except the ones in B_0. Indeed, if $j \in C \cup \{m-1\}$, by definition of f, we have

$$f(v_{i_j+k}) \equiv f(v_{i_j}) + 1 + 2k \equiv f(v_{i_{j+1}}) - 3 + 2k \pmod{2m}, \qquad (**)$$

except the second equality holds only for $j \in C$. Note that since $f(v_1) = 2b+1$, the pattern $\ll 2b-2, 2b-4, \dots, 2b \gg$ used, backwards, for vertices in B_0 (if $v_1 > 1$) is a formula similar to the last part in (**), that is, $f(v_{i_1-1+k}) \equiv f(v_{i_1}) - 3 + 2k \pmod{2m}$ for all k, $-(i_1 - 2) \le k \le 0$.

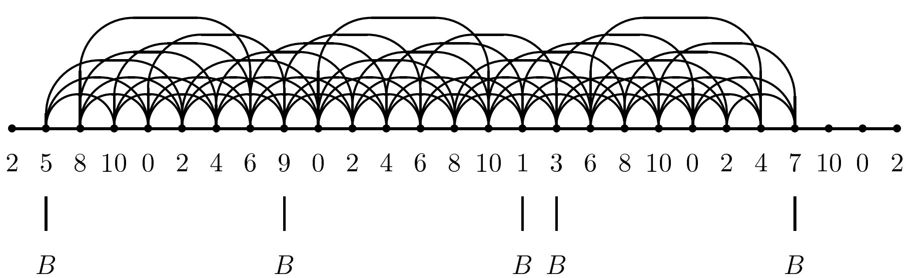

Figure 3: An example with $a = 1$, $b = 2$, $p = 1$, $\chi(G) = 6$ and $\mathrm{csp}_1(G) = 10$.

From the coloring f, one can observe that the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_{m-1}}$ receive distinct odd colors $\{1,3,5,\ldots,2m-3\}$, and other vertices receive even colors $\{0,2,4,\ldots,2m-2\}$. Since $C \neq \emptyset$ and for any $j \in C$, $i_{j+1} - i_j - 1 = k_j m \geq m$, so $\{f(v_{i_j+1}), f(v_{i_j+2}), \cdots, f(v_{i_{j+1}-1})\} = \{0,2,4,\cdots,2m-2\}$. Hence, f is onto.

Now it remains to show that f is an N-coloring. It suffices to claim that for any $uv \in E(G)$, neither one of the following two is possible:

(A) $f(u) = f(v) = 2t$ for some $0 \leq t \leq m-1$;

(B) $f(u) = 2t-1$ and $f(v) \in \{2t, 2t-2\}$ for some $1 \leq t \leq m-1$.

To show that (A) is impossible, suppose $f(u) = f(v) = 2t$ for some $0 \leq t \leq m-1$. Since u and v are adjacent, u and v together with all the vertices between them on P form a clique K. If u and v belong to the same block or if they belong to two non-consecutive blocks, then it is clear that $|K| \geq m+1$, a contradiction.

Now we assume that u and v belong to consecutive blocks. Without loss of generality, assume the ordering of u and v on P is u before v, and suppose $d_P(u,v)$ is the smallest. Let $u \in B_j$, then there are the following two cases:

Case A.1. $j \in \{0,1,\ldots,p-1,p+a+b,p+a+b+1,\ldots,m-2\}$: Then $v \in B_{i_{j+1}}$. Since $d_P(u,v)$ is the smallest, without loss of generality, we may assume $u = v_{i_j+lm+s}$ for some l so that $0 \leq s \leq m-1$ (if $j=0$, let $u = v_{i_1-1+(s-m)}$ for some s, $0 \leq s \leq m$), and $v = v_{i_{j+1}+q}$ for some $1 \leq q \leq m-1$. Hence, by definition of f, we have

$$\begin{aligned} 2t = f(v_{i_j+lm+s}) &= f(v_{i_j+s}) \equiv f(v_{i_{j+1}}) - 3 + 2s \pmod{2m} \\ &= f(v_{i_{j+1}+q}) \equiv f(v_{i_{j+1}}) + 1 + 2q \pmod{2m}. \end{aligned}$$

This implies that $q \equiv s-2 \pmod{\text{m}}$ and $|K| \geq (m-s+1)+1+(s-2) = m$. (Note that this also holds if $j=0$, since $f(v_{i_1-1+(s-m)}) \equiv f(v_{i_1}) - 3 + 2(s-m) \equiv f(v_{i_1}) - 3 + 2s \pmod{2m}$.) Hence K is a maximum clique, contradicting $v_{i_{j+1}} \in K \cap B$.

Case A.2. $j \in \{p, p+a\}$: Here we give the proof for $j = p$, the proof for $j = p+a$ can be obtained by a similar approach. Suppose $u \in B_p$, then $v \in B_{p+a}$. Since $d_P(u,v)$ is the smallest, without loss of generality, we may assume $u = v_{i_p+lm+s}$ for some l, $0 \leq s \leq m-1$, and $v = v_{i_{p+a}+q}$ for some $1 \leq q \leq m-1$. Then we have:

$$\begin{aligned} 2t &= f(v_{i_p+lm+s}) = f(v_{i_p+s}) = f(v_{i_p}) + 1 + 2s \\ &\equiv (2b+4p-3) + 1 + 2s \pmod{2m} \\ &\equiv 2s - 2a - 4 \pmod{2m} \quad (\text{since } \; 4p = 2(m-1-a-b)) \\ &= f(v_{i_{p+a}+q}) = f(v_{i_{p+a}}) + 1 + 2q \\ &\equiv (-3) + 1 + 2q \equiv 2q - 2 \pmod{2m}. \end{aligned}$$

Therefore, we have $q \equiv s-a-1 \pmod{m}$, so $|K| \geq (m-s+1)+a+(s-a-1)=m$, contradicting $v_{p+a} \in K \cap B$.

To show that (B) is impossible, suppose there exists $uv \in E(G)$ such that $f(u)=2t-1$ and $f(v) \in \{2t-2, 2t\}$ for some $1 \leq t \leq 2m-1$. Then $u = v_{i_j}$ for some $1 \leq j \leq m-1$. On P, the vertices between u and v together with u, v form a clique K. Because $u = v_{i_j} \in B$, $\mathrm{d}_P(u,v) < m-1$. We claim the following two possible cases:

Case B.1. $j \in \{1,2,\ldots,p,p+a+b+1,p+a+b+2,\ldots,m-1\}$: Since $d_P(u,v) < m-1$, one has $v \in B_j \cup B_{j-1}$. If $v \in B_j$, then $v = v_{i_j+s}$ for some $1 \leq s \leq m-2$. By definition of f, $f(v) = f(v_{i_j})+1+2s \equiv 2t+2s \pmod{2m} \in \{2t-2, 2t\}$. This implies $s \in \{0, m-1\}$, a contradiction. The proof for $v \in B_{j-1}$ is similar and we should omit it.

Case B.2. $j \in \{p+1,p+2,\ldots,p+a\}$ or $j \in \{p+a+1,p+a+2,\ldots,p+a+b\}$: We give a proof here for the case $j \in \{p+1,p+2,\ldots,p+a\}$, the proof for the case $j \in \{p+a+1,p+a+2,\ldots,p+a+b\}$ can be obtained by a similar process. Suppose $u = v_{p+k'}$ for some $1 \leq k' \leq a < m-1$. Then $v \in B_p \cup B_{p+a}$. By definition of f, $2t-1 = f(u) \equiv 2(k'-a)-3 \pmod{2m}$, so $f(v) \in \{2(k'-a)-4, 2(k'-a)-2\} \pmod{2m}$.

If $v \in B_p$, then $v = v_{i_p+lm+s}$ for some $1 \leq s \leq m-1$. Hence $f(v) = f(v_{i_p})+1+2s = 4p+2b-3+1+2s \equiv 2s-2a-4 \in \{2(k'-a)-4, 2(k'-a)-2\} \pmod{2m}$. Therefore, $s \in \{k', k'+1\}$. Because $i_{p+1} = i_p + k_p m + 1$ for some positive integer k_p, we have $d_p(u,v) \geq m-(k'+1)+k' = m-1$, a contradiction.

If $v \in B_{p+a}$, then $v = v_{i_{p+a}+s}$ for some $1 \leq s \leq m-2$. Hence, by definition of f, $f(v) = f(v_{i_{p+a}})+1+2s = 2s-2 \equiv 2(k'-a)-4$ or $2(k'-a)-2 \pmod{2m}$. Therefore, we have $s \equiv k'-a-1$ or $k'-a \pmod{m} = m+(k'-a-1)$ or $m+(k'-a)$. This implies $d_P(u,v) \geq a-k'+s = a-k'+m+(k'-a-1) = m-1$, a contradiction. $\square$

In the next three theorems, we give complete solutions for unit interval graphs with $\chi(G)=3$.

Theorem 10 *Suppose G is a unit interval graph on n vertices, $n \geq 5$, and $\chi(G)=3$. Then $\mathrm{csp}_1(G)=4$, if there exist $u,v \in B$, $u \neq v$, such that $uv \in E(G)$ or $d_P(u,v) \not\equiv 2 \pmod{3}$ on some compatible vertex ordering $P = v_1, v_2, \ldots, v_n$.*

Proof. If there exist $u,v \in B$ such that $uv \in E(G)$, then $d_P(u,v)=1$ for any compatible vertex ordering P, for otherwise u and v are contained in some maximum clique. Therefore, by Theorem 6, $\mathrm{csp}_1(G) = 2\chi(G)-2 = 4$.

Suppose there exist $u,v \in B$ such that $uv \notin E(G)$ and $d_P(u,v) \not\equiv 2 \pmod{3}$. If $d_P(u,v) \equiv 1 \pmod{3}$, by Theorem 9 with $a=b=1$, we have $\mathrm{csp}_1(G)=4$.

Suppose $d_P(u,v) \equiv 0 \pmod 3$. Let $u = v_i$ and $v = v_j$, then $i \equiv j \pmod 3$. Define the coloring f by:

$$f(v_{i+k}) = \begin{cases} 1, & \text{if } k = 0; \\ 3, & \text{if } k = j - i; \\ 4, & \text{if } k \equiv 1 \pmod 3,\ 1 \le k < j - i - 1; \\ 0, & \text{if } k \equiv 2 \pmod 3,\ 1 \le k < j - i - 1; \\ 2, & \text{if } k \equiv 0 \pmod 3,\ 1 \le k < j - i - 1; \end{cases}$$

for the vertices preceding v_i (if there is any), use the pattern $\ll 4,2,0 \gg$, backwards, and for the remaining vertices (if there is any), use the pattern $\ll 0,2,4 \gg$. It is easy to verify that f is an N-coloring for G, so $\mathrm{csp}_1(G) = 4$. □

To complete the family of unit interval graphs with $\chi(G) = 3$, it remains to consider the case that $|V(G)| > 5$ and B has exactly two vertices (for which we have the result below). If $|B| = 1$, by Theorems 4 and 5, $\mathrm{csp}_1(G) = 5$, if $n > 5$; and $\mathrm{csp}_1(G) = \infty$, otherwise. If B contains three vertices $v_a < v_b < v_c$ on P, then at least one of the pairs (v_a, v_b), (v_b, v_c) or (v_a, v_c) has distance $\not\equiv 2 \pmod 3$ on P, so $\mathrm{csp}_1(G) = 4$ by Theorem 10.

Theorem 11 *Suppose G is a connected unit interval graph with $\chi(G) = 3$, $|V(G)| = n > 5$, and $P = v_1, v_2, \cdots, v_n$ is a compatible vertex ordering. If $B = \{v_i, v_j\}$, where $j > i$ and $j - i \equiv 2 \pmod 3$. Then $\mathrm{csp}_1(G) = 5$ if and only if $v_{i+k}v_{i+k+2} \in E(G)$ for all $k \not\equiv 0 \pmod 3$ and $2 \le k \le j - i - 4$.*

Proof. ($\Rightarrow$) Assume $\mathrm{csp}_1(G) = 5$. Suppose to the contrary, $v_{i+k}v_{i+k+2} \notin E(G)$ for some $k \not\equiv 0 \pmod 3$ and $2 \le k \le j - i - 4$.

If $k \equiv 1 \pmod 3$, then define the coloring f by $f(v_i) = 1$, $f(v_j) = 3$; for vertices v_{i+1}, v_{i+2}, $\ldots$, v_{i+k+1}, repeat the pattern $\ll 4,2,0 \gg$ (i.e., $f(v_{i+1}) = 4$, $\cdots$, $f(v_{i+k}) = 4$, and $f(v_{i+k+1}) = 2$); for vertices v_{i+k+2}, v_{i+k+3}, $\ldots$, v_{j-1}, repeat the pattern $\ll 4,0,2 \gg$ (then $f(v_{j-1}) = 0$); for vertices preceding v_i, repeat the patter $\ll 4,2,0 \gg$ backwards; and for the vertices after v_j, repeat the pattern $\ll 0,2,4 \gg$ until the last vertex is colored. This gives an N-coloring for G with span 4, contradicting $\mathrm{csp}_1(G) = 5$.

If $k \equiv 2 \pmod 3$, then define the coloring f by $f(v_i) = 1$, $f(v_j) = 3$; for vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+k+1}$, repeat the pattern $\ll 4,0,2 \gg$ (i.e., $f(v_{i+k}) = 0$ and $f(v_{i+k+1}) = 2$); and for vertices $v_{i+k+2}, v_{i+k+3}, \ldots, v_{j-1}$, repeat the pattern $\ll 0,4,2 \gg$ (then $f(v_{j-1}) = 0$); for vertices preceding v_i, repeat the pattern $\ll 4,2,0 \gg$ backwards; and for the vertices after v_j, repeat the pattern $\ll 0,2,4 \gg$ until the last vertex is colored. This gives an N-coloring for G with span 4, a contradiction.

($\Leftarrow$) Suppose $v_{i+k}v_{i+k+2} \in E(G)$ for all $k \not\equiv 0 \pmod 3$ and $2 \le k \le j - i - 4$. Suppose $\mathrm{csp}_1(G) = 4$ and let $f : G \to \{0,1,2,3,4\}$ be an N-coloring for G. Then $f(v_i), f(v_j) \in \{1,3\}$ and $f(x) \in \{0,2,4\}$ for any

$x \neq v_i, v_j$, since $B = \{v_i, v_j\}$. Assume $f(v_i) = 1$ and $f(v_j) = 3$ (the proof for the case that $f(v_i) = 3$ and $f(v_j) = 1$ is similar), then $f(v_{i+1}) = 4$. Because G is connected, $v_l v_{l+1} \in E(G)$ for all $1 \leq l \leq n-1$. Combining this with the assumption that $v_{i+k}v_{i+k+2} \in E(G)$ for all $k \not\equiv 0 \pmod 3$ and $1 \leq k \leq j-i-3$ (Since $v_{i+1}, v_{j-1} \in A$ and $v_i, v_j \in B$, we have $v_{i+1}v_{i+3}, v_{j-3}v_{j-1} \in E(G)$). One must have $f(v_{i+x}) = 4$ for all $x \equiv 1 \pmod 3$, $1 \leq x \leq j-i-1$, implying that $f(v_{j-1}) = 4$, contradicting $f(v_j) = 3$. □

In conclusion, we have

Theorem 12 *Suppose G is a connected unit interval graph on n vertices and $\chi(G) = 3$. Let $P = v_1, v_2, \cdots, v_n$ be a compatible vertex ordering of G. Then*

$$\mathrm{csp}_1(G) = \begin{cases} \infty, & \text{if } n < 5, \text{ or } n = 5 \text{ and } |B| = 1; \\ 5, & \text{if } n > 5 \text{ and } |B| = 1, \text{ or } n > 5, B = \{v_i, v_j\}, \text{ where} \\ & \quad j > i,\ j - i \equiv 2 \pmod 3, \text{ and } v_{i+k}v_{i+k+2} \in E(G) \\ & \quad \text{for all } k \equiv 0 \pmod 3 \text{ and } 2 \leq k \leq j-i-4; \\ 4, & \text{otherwise.} \end{cases}$$

Acknowledgment. The authors are grateful to the referee for an immediate report.

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