PHILOSOPHIA MATHEMATICA

PHILOSOPHY OF MATHEMATICS,
ITS LEARNING, AND ITS APPLICATION

PHILOSOPHIE DES MATHÉMATIQUES,
LEURS ÉTUDE ET APPLICATIONS

Offprint

Non-Uniqueness as a Non-Problem

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THE PHILOSOPHICAL JOURNAL OF
THE CANADIAN SOCIETY FOR HISTORY
AND PHILOSOPHY OF MATHEMATICS

LA SOCIÉTÉ CANADIENNE D'HISTOIRE ET
DE PHILOSOPHIE DES MATHÉMATIQUES

SERIES III

Non-Uniqueness as a Non-Problem†

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1. The Non-Uniqueness Objection to Platonism

Let *mathematical platonism* be the view that our mathematical theories are descriptions of abstract mathematical objects, *i.e.*, non-spatio-temporal mathematical objects. It is widely believed that there are two important problems with this view, both of which were made famous by Paul Benacerraf. One, the epistemological problem, is suggested by Benacerraf's 1973 paper, 'Mathematical truth', and the other, the non-uniqueness problem, or the multiple-reductions problem, is suggested by his 1965 paper, 'What numbers could not be'. I have argued elsewhere that platonists can solve the epistemological problem.[1] In this paper, I will argue that they can also solve the non-uniqueness problem; or to be more precise, I will argue that there is, in fact, not really a problem here at all.

In a nutshell, the non-uniqueness problem is this: platonism suggests that our mathematical theories describe *unique* collections of abstract objects, but in point of fact, this does not seem to be the case. But let me spell this argument out in more detail. I will state the argument in terms of arithmetic, as is commonly done, but it should be noted that analogous arguments can be given in connection with other mathematical theories. The argument proceeds as follows:

(1) If there are any sequences of abstract objects which satisfy the axioms of Peano Arithmetic (PA), then there are infinitely many such sequences.

(2) There is nothing 'metaphysically special' about any of these sequences

† Earlier versions of this paper were presented at UCLA and at the 1996 meeting of the Association of Symbolic Logic at the University of Wisconsin, Madison. I received helpful comments at these talks from several people, including Geoffrey Hellman, Charles Parsons, Jody Azzouni, Colin McLarty, Joseph Almog, Tony Martin, Yiannis Moschovakis, and Paul Hovda. I would like to thank these people, as well as Penelope Maddy, Michael Resnik, and Colin Cheyne, who provided comments on an earlier draft, and the National Endowment for the Humanities, which provided funding.

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[1] See my [1995] and chapter III of my [forthcoming]. The basic idea behind the solution will emerge below.

that makes it stand out from the others as *the* sequence of natural numbers.

Therefore,

(3) There is no unique sequence of abstract objects which is the natural numbers.

But

(4) Platonism entails that there *is* a unique sequence of abstract objects which is the natural numbers.

Therefore,

(5) Platonism is false.

(I said above that this argument is suggested by Benacerraf's paper, but I should point out that it is not explicitly formulated there. The sub-argument in (1)–(3) is pretty clearly contained in that paper, but (4) is not. Now, most commentators seem to think that this premise is more or less implicit in Benacerraf's paper, but it seems to me equally likely that Benacerraf never meant to commit to (4)—that he only meant to establish (3) and not (5). In any event, I will not be concerned with this exegetical issue here. I am simply going to respond to the argument in (1)–(5).[2])

It seems to me that the only vulnerable parts of the non-uniqueness argument are (2) and (4). The two inferences—from (1) and (2) to (3) and from (3) and (4) to (5)—are both fairly trivial. Moreover, as we will see, (1) is virtually undeniable. (And it should be noted that we cannot make (1) any less trivial by taking PA to be a second-order theory and, hence, categorical. This will only guarantee that all the models of PA are isomorphic to one another. It will not deliver the desired result of there being only one model of PA.) So it seems that platonists have to attack either (2) or (4). In what follows, I will attack (4). But before I do that, I want to say a few words about the strategy of attacking (2).

2. Trying to Salvage the Numbers

I begin by sketching Benacerraf's argument in *favor* of (2). He proceeds here in two stages: first, he argues that no sequence of *sets* stands out as *the* sequence of natural numbers, and second, he extends the argument so that it covers sequences of other sorts of objects as well. The first claim, *i.e.*, the claim about sequences of sets, is motivated by reflecting on the numerous set-theoretic reductions of the natural numbers. Benacerraf concentrates, in particular, on the reductions given by Zermelo and von Neumann. Both of these reductions begin by identifying 0 with the null set, but Zermelo identifies $n + 1$ with the singleton $\{n\}$, whereas von Neumann identifies

[2] The thesis that Benacerraf was explicitly attacking was the thesis that arithmetic is about abjects. I will not go into this, but the present paper provides a defense of this thesis, as well as a defense of platonism, because the view defended against Benacerrafian considerations here is a version of object-platonism.

$n+1$ with the union $n \cup \{n\}$. Thus, the two progressions proceed like so:

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$$

and

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$$

Benacerraf argues very convincingly that neither of these progressions provides a better reduction than the other, that there is no non-arbitrary reason for identifying the natural numbers with one of these sequences rather than the other or, indeed, with any of the many other set-theoretic sequences that would seem just as good here, *e.g.*, the sequence that Frege suggests in his reduction.

Having thus argued that no sequence of sets stands out as *the* sequence of natural numbers, Benacerraf extends the point to sequences of other sorts of objects. His argument here proceeds as follows. From an arithmetical point of view, the only properties of a given sequence that *matter* to the question of whether it is the sequence of natural numbers are *structural* properties. In other words, nothing about the individual objects in the sequence matters—all that matters is the structure that the objects jointly possess. Therefore, any sequence with the right structure will be as good a candidate for being the natural numbers as any other sequence with the right structure. In other words, any *ω-sequence* will be as good a candidate as any other. Thus, we can conclude that no one sequence of objects stands out as *the* sequence of natural numbers.

It seems to me that if Benacerraf's argument for (2) can be blocked at all, it will have to be at this second stage, for I think it is more or less beyond doubt that no sequence of *sets* stands out as *the* sequence of natural numbers. So how can we attack the second stage of the argument? Well, one strategy here is to argue that all Benacerraf's argument shows is that the natural numbers cannot be *reduced* to anything else. On this view, the first part of the argument shows that they cannot be reduced to sets, and the second part shows that they cannot be reduced to anything else. But the possibility remains that numbers are simply irreducible entities. In other words, it may be that while there *are* numerous sequences of sets and functions and properties that satisfy PA, there is also a sequence of irreducible natural *numbers* that satisfies PA. And if there is such a sequence, then it certainly does stand out from all other sequences as *the* sequence of natural numbers. (Michael Resnik has made a similar point. He maintains that while Benacerraf has shown that numbers are not sets or functions or chairs, he has not shown that numbers are not objects, because he has not shown that numbers are not *numbers*.[3])

[3] Resnik [1980], p. 231.

Now, one problem with this response is that Benacerraf did not formulate the second stage of his argument for (2) in terms of *reductions* at all. The first stage of his argument was formulated in terms of reductions, but the second stage was not; it was based upon the observation that only structural facts about a sequence of objects are relevant to the question of whether that sequence is the sequence of natural numbers. But one might think that we can preserve the *spirit* of the response given in the last paragraph—*i.e.*, the one formulated in terms of reductions—while responding more directly to the argument that Benacerraf actually used. In particular, one might try to do this in something like the following way.

'There is some initial plausibility to Benacerraf's claim that only structural facts are relevant to the question of whether a given sequence of objects is the sequence of natural numbers. For (a) only structural facts are relevant to the question of whether a given sequence is *arithmetically adequate*, *i.e.*, whether it satisfies PA; and (b) since PA is our best theory of the natural numbers, it would seem that it captures *everything we know* about those numbers. But a moment's reflection reveals that this is confused, that PA does *not* capture everything we know about the natural numbers. There is nothing in PA that tells us that the number 17 is not the inventor of Cocoa Puffs, but nonetheless, we know (pre-theoretically) that it is not. And there is nothing in PA that tells us that numbers are not sets, but again, we know that they are not. Likewise, we know that numbers are not functions or properties or chairs. Now, it is true that these facts about the natural numbers are not *mathematically important*—that is why none of them are included in PA—but in the present context, that is irrelevant. What matters is this: while Benacerraf is right that if there are any sequences of abstract objects that satisfy PA, then there are many, the same cannot be said about our *full conception of the natural numbers*, or FCNN. We know, for instance, that no sequence of sets or functions or chairs satisfies FCNN, because it is built into our conception of the natural numbers that they do not have members, that they cannot be sat on, and so forth. Indeed, we seem to know that no sequence of things that are not natural numbers satisfies FCNN, because part of our conception of the natural numbers is that they are natural numbers. Thus, it seems that we know of only *one* sequence that satisfies FCNN, *viz.*, the sequence of natural numbers. But, of course, this means that (2) is false, that one of the sequences that satisfies PA stands out as *the* sequence of natural numbers.'

Before saying what I think is really wrong with this argument, let me set a couple of worries aside. First, one might be a bit uneasy about the appeal to FCNN here; one might think that before platonists can rely upon any claims about FCNN, they need to say exactly what this 'theory' (or whatever it is) consists in. Does it have axioms and theorems? Is it first-order? Is it second-order? Can it be formalized at all? And so on. It seems to me, however, that the appeal to FCNN is rather uncontroversial and that to ask these sorts of questions is to miss the point of this appeal. There are, of course, controversial questions about what exactly is contained in FCNN and what is not; but the basic idea behind FCNN—that we have beliefs about the natural numbers that are not captured by PA—seems relatively uncontroversial. Just about everyone agrees, for instance, that the number 3 is non-red; even mathematical fictionalists, who deny that there is any such thing as 3, agree that non-redness is built into our 'conception of 3'. Now, I do not mean to suggest that *no one* would take issue with the idea of FCNN; but it seems to me that there is nothing problematic about *platonists* appealing to FCNN, for the idea here goes hand-in-hand with the platonistic conception of mathematics. Moreover, given that we are allowing platonists to speak of FCNN, it would seem inappropriate to demand that they get very precise about its nature. FCNN is just the collection of everything that we, as a community, believe about the natural numbers (and everything that follows from those beliefs). It is not a formal theory, and so it is not first-order or second-order, and it does not have any axioms in anything like the normal sense. Moreover, it is likely that there is no clear fact of the matter as to precisely which sentences are contained in FCNN (although for *most* sentences, there *is* a clear fact of the matter—*e.g.*, '3 is prime' and '3 is not red' are clearly contained in FCNN, whereas '3 is not prime' and '3 is red' are clearly not). So it seems to me that to get precise about FCNN would do violence to the very idea of FCNN.

A second worry that one might have about the argument of the paragraph before last concerns the *details* of FCNN. More specifically, one might grant the general idea of FCNN but take issue with the particular claim that it is built into FCNN that numbers are not, *e.g.*, sets or properties. But I do not think there is much to motivate this; it seems pretty clear to me that it *is* built into FCNN that numbers are not sets or properties (although I should note here that nothing I am going to say will depend on this claim).

So what *is* wrong with the above response to the non-uniqueness argument? In a nutshell, the problem is that this response begs the question against Benacerraf, because it simply helps itself to 'the natural numbers'. We can take the point of Benacerraf's argument to be that if all the ω-sequences were, so to speak, 'laid out before us', we could have no good reason for singling one of them out as *the* sequence of natural numbers. Now, the above response does show that the situation here is not as grim as Benacerraf has made it seem, because it shows that *some* ω-sequences can be ruled out as definitely *not* the natural numbers. In particular, any ω-sequence that contains an object that we recognize as a non-number—*e.g.*, a function or a chair or (on the view in question here) a set—can be ruled out in this way. In short, any ω-sequence that does not satisfy

FCNN can be so ruled out. But platonists are not in any position to claim that all ω-sequences but one can be ruled out in this way; for since they think that abstract objects exist *independently of us*, they must admit that there are very likely numerous kinds of abstract objects that we have never thought about and, hence, that there are very likely numerous ω-sequences that satisfy FCNN and only differ from one another in ways that no human being has ever imagined. I do not see any way for platonists to escape this possibility, and so it seems to me very likely that (2) is true and, hence, that (3) is also true.[4]

There are, of course, various responses that platonists might make to this argument, but I do not want to pursue this any further, because the solution that I am going to give (in section 4) to the non-uniqueness problem does not depend upon the attitude I have taken here towards (2) and (3). That is, my response is consistent with the *falsity* of (2) and (3), although I will be assuming from here on out that they are true.[5]

3. Structuralism

If (2) is true, then platonists must either abandon their view or else find some plausible way to reject (4). Now, there is one very famous way of rejecting (4) which, I think, does *not* provide an adequate solution to the non-uniqueness problem. The idea here is to adopt a platonistic version of Benacerraf's own view, *i.e.*, a platonistic version of *structuralism*.[6] On this view, arithmetic is about not some particular sequence of objects, but rather, the structure that all ω-sequences have in common. The reason this view is consistent with *platonism* is that structures can be taken to be real abstract entities, existing independently of us and outside of spacetime.

[4] One might worry that a similar argument could be constructed in an *empirical* context. But such an argument would not go through. For we have *perceptual contact* with physical objects, and this serves to ground our concrete singular terms to *particular* objects in a way that our abstract singular terms are *not* tied to particular objects. Now, I am not suggesting here that a mere appeal to perceptibility eliminates all worries about uniqueness in connection with physical-object talk. To name just one problem here, the appeal to perceptibility will not do anything to block Quine-style inscrutability-of-reference worries. My point is simply that the appeal to perceptibility blocks *the worry in the text* from applying in physical-object cases. To see that this is so, consider an attempt to apply this worry in such a case. One might begin here by, *e.g.*, telling us to imagine a twin-earth version of Bill Clinton who is just like our Bill Clinton in all ways that we have ever considered but different in some way that we have never thought about. Now, I think it is dubious that such a creature could really be just like our Bill Clinton in *all* ways that we have ever considered, but even if we assume that this is possible, it should be clear that our term 'Bill Clinton' does not refer to any such creature, because no such creature is appropriately 'connected' to us.

[5] I am going to defend a version of platonism that embraces non-uniqueness, and so it might seem as though my response is *not* consistent with the falsity of (2) and (3). But my real point is going to be that platonists can *allow* non-uniqueness, not that they have to *commit* to it.

[6] See, *e.g.*, Resnik [1981] and [1997] and Shapiro [1989] and [1997].

The trouble with this response to the non-uniqueness problem is not that structuralism is false; it's that it does not solve the problem. This can be appreciated by merely noting that we can reformulate the non-uniqueness argument in (1)–(5) so that it applies to structuralism as well as to object-platonism. We can do this as follows:

(1′) If there are any parts of the mathematical realm that satisfy the axioms of PA, then there are infinitely many such parts.

(2′) There is nothing 'metaphysically special' about any of these parts of the mathematical realm that makes it stand out from the others as *the* sequence of natural numbers (or natural-number positions or whatever).

Therefore,

(3′) There is not a unique part of the mathematical realm which is the sequence of natural numbers (or natural-number positions or whatever).

But

(4′) Platonism entails that there *is* a unique part of the mathematical realm which is the sequence of natural numbers (or natural-number positions or whatever).

Therefore,

(5′) Platonism is false.

So platonists cannot solve the non-uniqueness problem by merely adopting structuralism and rejecting the thesis that mathematics is about objects, because the problem still remains even after we make the switch to a structuralistic platonism.

But we cannot yet close the book on structuralism. Before we do that, we need to ask whether there is any way of responding to this new version of the argument that is available to structuralists but not to object-platonists. More specifically, we need to ask whether there is any way of refuting (2′) that is available to structuralists but not to object-platonists, because this is how structuralists will want to respond to the new version of the argument. (It might seem surprising that structuralists would respond in this way, since, as we have seen, they respond to the original version of the argument by rejecting (4). But if we think of the strategies of rejecting (2′) and (4′) as the strategies of trying to *salvage* uniqueness and *abandon* uniqueness, respectively, then it makes perfectly good sense to think of structuralists as rejecting (2′). For despite their rejection of (4), they do not want to abandon uniqueness. They reject (4) because of the role played in that sentence by the word 'object'—not because of the role played by the word 'unique'. As far as uniqueness is concerned, structuralists want to *salvage* it: their claim is that arithmetic is about *the* structure that all ω-sequences have in common. Or at any rate, this is the *standard* structuralist view.[7]

[7] Actually, I should say that this is how *I interpret* the standard structuralist view. The

Of course, the possibility remains that *some* structuralists might want to respond to the non-uniqueness problem by *abandoning* uniqueness, *i.e.*, by rejecting (4′). I will say a few words about this below, but first, I want to discuss (2′); in particular, I want to argue that structuralists are no better off with respect to the strategy of rejecting (2′) than object-platonists are.)

It seems to me very unlikely that there is a plausible way of refuting (2′) that is available to structuralists but not to object-platonists. For the claim that there is a unique structure—*i.e.*, sequence of positions—that stands out as *the* sequence of natural numbers seems just as implausible as the claim that there is a unique sequence of *objects* that stands out as *the* sequence of natural numbers. And, indeed, it seems implausible for the same reason: since structures exist independently of us in an abstract mathematical realm, it seems very likely that there are numerous things in the mathematical realm that count as structures, that satisfy FCNN, and that differ from one another only in ways that no human being has ever imagined. (And this, by the way, shows not just that structuralists are as badly off as object-platonists with respect to (2′), but also that (2′) is just as well motivated as (2). In other words, the point here is that (2′) can be motivated in the same way that (2) was motivated in the last section.)

Structuralists might try to respond here by arguing for

(A) There exists a *unique* structure—or sequence of positions—that satisfies FCNN

in something like the following way. 'There is no more *to* a structure than the relations that hold between its positions. That is, positions do not have any properties other than those they have in virtue of the relations that they bear to other positions in the structure. Therefore, any two structures that are structurally equivalent, *i.e.*, isomorphic, are identical with one another.[8] But it seems safe to assume that any two structures that satisfy FCNN are isomorphic to one another, *i.e.*, that FCNN is categorical, *i.e.*, that non-standard models of arithmetic do not satisfy FCNN. Therefore, it

reason I add this qualification is simply that, to the best of my knowledge, no structuralist has ever explicitly discussed this point. This, I think, is a bit puzzling, because one of the standard arguments for structuralism is supposed to be that it provides a way of avoiding the non-uniqueness problem. I suppose that structuralists just have not noticed that there are general versions of the non-uniqueness argument that apply to their view as well as to object-platonism. They seem to think that the non-uniqueness problem just disappears as soon as we adopt structuralism.

[8] I should note here that Resnik ([1997], chapter 10) explitly *rejects* the idea that isomorphism is the identity condition for structures. His view here is a bit 'slippery', because he does not want to commit to the claim that structures are *entities*, but, ignoring this point for the moment, he allows that two structures can be isomorphic but distinct. I think that this view is superior to the view that isomorphic structures are identical, for as we shall see, I think that the latter view is teetering on unintelligibility. But because Resnik takes this line, I do not think he has any response to the argument of the last paragraph.

seems to follow that there is only *one* structure that satisfies FCNN.'

I will argue in a moment that this is not a good argument for (A). But the more important point to make here is that even if structuralists could motivate (A), this would not provide them with an adequate response to the non-uniqueness problem. They would also need to motivate

(B) The unique sequence of positions mentioned in (A) stands out from all sequences of *objects* as *the* sequence of natural numbers.

But (B) seems very implausible. For even if (A) were true, there would still be numerous sequences of objects that satisfied FCNN, and these sequences would all have as much claim to being the natural numbers as the sequence of positions mentioned in (A) would have, and so the non-uniqueness problem would still remain. I do not see any plausible way for structuralists to avoid this. I suppose they might try to claim that it is built into FCNN that there is no more to the numbers than the relations that hold between them, but (a) it is totally unclear how they could *justify* this claim, and (b) the claim just seems entirely implausible. Indeed, if anything, we have reason to think that FCNN *rules out* the idea that there is no more to the numbers than the relations that hold between them, so that the structuralist sequence mentioned in the last paragraph is not even one among many viable candidates for being the sequence of natural numbers. The reason I say this is that (a) it seems to be built into FCNN that numbers are *non-spatio-temporal* (or if you doubt this, it is surely built into FCNN that numbers are *non-red*); but (b) properties like these have nothing to do with the relations that hold between the numbers. Moreover, to take this argument one step further, it seems that the property of having only structural properties is *itself* a non-structural property, and so the notion of structure at work in the last paragraph seems to be incoherent.

These remarks also serve to undermine the structuralist's argument for (A). That argument was based on the idea that structures have *no* non-structural properties, but as we have just seen, it is doubtful that there is *anything* that has no non-structural properties. (Without the claim that structures have no non-structural properties, the above argument for (A) would not go through, because it would then be possible for there to be numerous structures that satisfy FCNN and differ from one another only in non-structural ways that we have never imagined. The whole point of the above argument was to rule out this possibility by not allowing structures to have non-structural properties that we have never imagined.)[9]

[9] Lurking in the background here is an argument against structuralism. It seems true that

(S) All mathematically important facts are structural facts.

But to affirm (S) is not yet to endorse structuralism, for object-platonists can easily account for (S): they can simply claim that all mathematically important facts concern the relations between mathematical objects. Structuralists differ from object-platonists

I conclude, then, that as was the case with (2) and (3), it is very likely that (2′) and (3′) are true. And so I also conclude that platonists cannot solve the non-uniqueness problem by merely adopting structuralism. (Actually, this second conclusion has not really been established yet, because the possibility still remains that structuralism could provide platonists with a solution to the non-uniqueness problem via the strategy of rejecting (4′). We are going to see in the next section, however, that *all* platonists can solve the problem by rejecting (4′), regardless of whether they endorse structuralism. I am going to develop this solution in object-platonist terms, but one could also develop a structuralist version of the solution. But even if we did this, structuralism would not play any *role* in the solution; that is, the solution would still be the same—it would just be stated in structuralist terms. Thus, what we can say is this: the arguments of this section and the next, taken together, show that the question of whether platonists endorse structuralism is wholly irrelevant to the question of whether they can solve the non-uniqueness problem.[10])

I end with the same remark I made at the end of section 2: while there are certainly responses that one might offer to the argument of this section, I do not want to pursue any of them here, because the solution that I will give (in the next section) to the non-uniqueness problem does not depend

in that they want to go beyond (S) and endorse an ontological or metaphysical claim. But *what* claim? Well, as a first shot, one might take the claim to be that *mathematics is about positions in structures*, as opposed to objects. But an obvious response to this is that positions in structures *are* objects. (Given that we refer to them with singular terms and quantify over them in first-order languages, why *wouldn't* they be objects?) So if structuralism is to be genuinely distinct from object-platonism, we need an account of how positions differ from other sorts of objects. One response to this problem—advocated by Parsons ([1990], pp. 303–304)—is to maintain that positions have no more to them than the relations they bear to other positions in the same structure. But this is precisely the sort of view that I have argued is incoherent. For whatever it is worth, I doubt that structuralists can meet this challenge. That is, I doubt that there is any important difference between the structuralist conception of mathematical objects as 'positions' and the traditional conception of mathematical objects. (Resnik has suggested to me that perhaps the difference lies in the fact that object-platonists often see facts of the matter where structuralists do not. But in the next section, I am going to introduce a *pluralistic* version of object-platonism that is not committed to the sorts of fact-of-the-matter claims that Resnik has in mind here.)

[10] Actually, I think this point can be generalized: the question of whether platonists endorse structuralism is irrelevant to the question of whether they can solve *any* of the problems with their view. Now, I do not have the space to justify this claim completely here, but I am going to come close. For aside from the non-uniqueness problem, the most important problem with platonism is the epistemological problem, and in the next section, I am going to describe very briefly how object-platonists can solve this problem (and once again, while I will state the solution in object-platonist terms, the same strategy of response could be employed by structuralists). In any event, if the above general claim about structuralism and platonism is correct, then it completely undermines structuralism, because the whole motivation for that view is that it is supposed to make the various problems with platonism—in particular, the non-uniqueness problem and the epistemological problem—more tractable.

upon the cogency of my refutation of the structuralist solution.

4. The Solution

Having dispensed with the idea that structuralism might provide us with a solution to the non-uniqueness problem, I want to go back to speaking in terms of the argument in (1)–(5) and forget about the argument in (1′)–(5′). I do this merely for reasons of elegance: it is simply less cumbersome to speak in terms of mathematical *objects* than parts of the mathematical realm.

In any event, it seems to me that the only remaining platonist strategy for responding to the non-uniqueness argument is to reject (4). That is, platonists have to give up on uniqueness. And they have to do this in connection with not just arithmetical theories like PA and FCNN, but all of our mathematical theories. They have to claim that while such theories truly describe collections of abstract mathematical objects, they do not pick out *unique* collections of such objects. (Or at the very least, platonists have to claim that if any of our mathematical theories does describe a unique collection of abstract objects, it is only by blind luck that it does.)

Now, this stance certainly represents a departure from traditional versions of platonism. But it cannot be seriously maintained that in making this move, we *abandon* platonism. For since the core of platonism is the belief in abstract objects—and since the core of mathematical platonism is the belief that our mathematical theories truly describe such objects—it follows that the above view is a version of platonism. Thus, the only question is whether there is some reason for thinking that platonists cannot make this move, *i.e.*, for thinking that platonists are *committed* to the thesis that our mathematical theories describe unique collections of mathematical objects. In other words, the question is whether there is any *argument* for (4)—or for a generalized version of (4) which holds of not just arithmetic but all of our mathematical theories.

It seems to me—and this is the central claim of this paper—that there is *not* such an argument. First of all, Benacerraf did not give any argument at all for (4).[11] Moreover, to the best of my knowledge, no one else has ever argued for it either. But the really important point here is that, *prima facie*, it seems that there could not *be* a cogent argument for (4)—or for a generalized version of (4)—because, on the face of it, (4) and its generalization are both highly implausible. The generalized version of (4) says that

[11] Actually, while it is clear that Benacerraf's 1965 paper does not address (4), one might maintain that there is something like an argument for (4) implicit in his 1973 argument for the claim that we ought to use the same semantics for Mathematese that we use for ordinary English. I will respond to this below.

(P) Our mathematical theories truly describe collections of abstract mathematical objects

entails

(U) Our mathematical theories truly describe *unique* collections of abstract mathematical objects.

This is a *really* strong claim. And as near as I can tell, there is absolutely no reason to believe it. Thus, it seems to me that platonists can simply accept (P) and reject (U). Indeed, they can endorse (P) together with the *contrary* of (U); that is, they can claim that while our mathematical theories do describe collections of abstract objects, none of them describes a unique collection of such objects. In short, platonists can avoid the so-called non-uniqueness 'problem' by simply *embracing* non-uniqueness, *i.e.*, by adopting *non-uniqueness platonism*, or NUP.

One might respond here as follows. 'Your argument is too quick. You are right that in order to really show that platonists *cannot* endorse NUP, we would have to provide an argument for (4) (and/or its generalization). And you are also right that the entailment claims inherent in (4) and its generalization are extremely implausible. But we might be able to grant that (4) and its generalization are false—*i.e.*, that (P) does not *entail* (U)—but still maintain that those who endorse (P) ought to endorse (U) as well. In other words, we might be able to grant that NUP is *intelligible* but still maintain that platonists ought not to endorse it. For it may be that NUP can be refuted, or made to seem implausible, on independent grounds.'

This worry is related to a second worry, namely, that the adoption of NUP is an *ad hoc* device, *i.e.*, that the only reason for platonists to endorse NUP is that it solves the non-uniqueness problem. In the remainder of this section, I will respond to these two worries. I will respond to the second worry by arguing that there are *independent* reasons for favoring NUP over traditional U-platonism, *i.e.*, traditional versions of platonism that endorse (U). And I will respond to the first worry by arguing the opposite point, *i.e.*, that there are no good reasons for favoring U-platonism over NUP.

In connection with the second worry, I think there are *several* independent reasons for favoring NUP over U-platonism. The most important is that platonists need to adopt NUP in order to respond to Benacerraf's *other* argument against platonism, *i.e.*, his epistemological argument. In a nutshell, the epistemological argument is that platonism could not be true, because (a) if it were true, then we would have knowledge of entities that exist outside of spacetime, and (b) we could not have such knowledge, because we exist wholly within spacetime and, hence, have no *access* to anything existing outside of spacetime. I have argued elsewhere that platonists can solve this problem if (and only if) they adopt what I call *full-blooded platonism*, or FBP.[12] FBP is (roughly) the view that all the mathematical objects that (logically) possibly *could* exist actually *do* exist. Now, I cannot give the complete justification here for the claim that FBP-ists can solve the epistemological problem, but the basic idea is as follows. If FBP is true, then every consistent purely mathematical theory truly describes some collection of abstract mathematical objects. Thus, to acquire knowledge of abstract objects, all we need to do is acquire knowledge that some purely mathematical theory is consistent. (It might be objected here that one would also need to know that FBP is true; but this objection reduces to a *skeptical* demand for an *internalist* epistemology, and I argue that in the present context, this demand is illegitimate.) But knowledge of the consistency of a mathematical theory—or any other kind of theory for that matter—does not require any sort of *access* to the objects that the theory is about.[13] Thus, the Benacerrafian lack-of-access objection has been answered: we can acquire knowledge of abstract objects without the aid of any sort of access to such objects, *i.e.*, without having any information-gathering *contact* with such objects.

Now, it should be clear that FBP is a version of NUP, *i.e.*, that FBP-ists would reject (U). For if the mathematical realm is as populated as FBP would have it, then it seems extremely unlikely that any of our mathematical theories—even those which include pre-theoretical claims, *e.g.*, FCNN—are uniquely satisfied. To take the example of FCNN, it seems overwhelmingly likely that there are numerous ω-sequences that satisfy FCNN and differ from one another only in ways that no human being has ever thought about. Thus, while FBP does not *entail* NUP, it seems that once we adopt FBP, we ought to adopt NUP as well. And so it seems to me that platonists have independent reason to favor NUP over U-platonism: they need to endorse NUP in order to solve Benacerraf's epistemological problem.[14]

[12] See my [1995] and chapter III of my [forthcoming].

[13] The point of this last claim is that knowledge of consistency does not involve knowledge of abstract objects. But one might respond as follows: 'While knowledge of the consistency of a mathematical theory does not depend upon any prior knowledge of the abstract objects that the theory in question is about, it might very well involve knowlededte of *other* abstract objects, *e.g.*, models or derivations.' This remark would be correct if I were speaking here of semantic or syntactic consistency, because they are platonistic notions that involve models and derivations, respectively. But I have in mind here an anti-platonistic notion of consistency, in particular, a primitive notion similar to the one discussed by Kreisel [1967]. (For discussions of this primitive anti-platonistic notion of consistency, see chapter III of my [forthcoming] and Field [1991].)

[14] Again, this is only the most important independent reason to favor NUP over U-platonism. There are also other reasons. For just as FBP is the only version of platonism that can account for the fact that we have mathematical knowledge, it is also the only version of platonism that can account for various other facts about mathematical practice—*e.g.*, the fact that axiom candidates can be justified for pragmatic reasons. I discuss these other reasons for favoring FBP-NUP over U-platonism in my [1995] and in chapter III of my [forthcoming].

(Of course, this argument relies upon the premise that *non*-full-blooded versions of platonism *cannot* solve the epistemological problem. Unfortunately, I do not have the space to justify this premise here, but I have done this elsewhere.[15])

So we seem to have good independent reasons for favoring NUP over traditional U-platonism. But I still need to argue—in order to block the *first* worry discussed above—that there are no good reasons for favoring U-platonism over NUP or FBP-NUP. To this end, I will consider what I think are the two most obvious and promising arguments that U-platonists might attempt here and show that they are not cogent.

The first argument that U-platonists might give for favoring their view over NUP, or FBP-NUP, can be put as follows. 'We should take our mathematical theories as being about unique collections of mathematical objects, because we use *singular terms* in these theories, and this would only be acceptable if there were unique referents for these terms. After all, if a singular term does not have a unique referent, we are inclined to say that it does not refer at all, that it suffers some sort of reference *failure*.'

It seems to me that the central premise here—the claim that the use of mathematical singular terms would only be acceptable if there were unique referents for these terms—requires *argument*. For this is precisely what NUP-ists deny. To deny that our mathematical theories are descriptions of unique collections of mathematical objects just *is* to deny that our mathematical singular terms have unique referents. Moreover, NUP-ists do not just deny that our mathematical singular terms have unique referents and leave it at that. They have a story to tell about why this is acceptable, or unproblematic. The reason, in a nutshell, is that the internal properties of mathematical objects are mathematically unimportant. As structuralists are quick to point out, all mathematically important facts are relational facts, *i.e.*, facts about the relations between mathematical objects, as opposed to facts about the internal properties of mathematical objects.[16] Because of this, it simply doesn't *matter* if our mathematical theories fail to pick out unique collections of objects (or if our mathematical singular terms fail to pick out unique referents), because we can capture the structural facts that we are really after *without* picking out unique collections of objects (or unique referents).

So the first point to note here is that NUP-ists have a reason for thinking that it would be acceptable to use singular terms in mathematics without there being unique referents for those terms. But we ought to ask whether U-platonists have any argument for the opposite conclusion. Is there any

[15] See chapter II of my [forthcoming].

[16] It might seem that the thesis that all mathematically important facts are structural facts could be used to motivate structuralism. But it cannot; as we saw in footnote 9, object-platonism is compatible with this thesis.

reason for thinking that it would be *unacceptable* to use singular terms in mathematics without there being unique referents for those terms? Or ignoring the issue of acceptability, is there any reason for thinking that, in point of actual fact, our mathematical singular terms do have unique referents? I can think of three arguments that U-platonists might offer here. They can be formulated as follows.

Argument 1: In abandoning unique reference, platonists abandon the ability to adopt a *standard semantics* for Mathematese, *i.e.*, a semantics that parallels the one we use for ordinary discourse. But the ability to adopt such a semantics for Mathematese has always been one of the main motivations for mathematical platonism.

Argument 2: Mathematicians seem to have unique objects *in mind* when they use singular terms. Indeed, considerations of this sort can be used to argue against NUP directly, for it also seems that mathematicians have unique *collections* of objects in mind when they construct their *theories*, *e.g.*, arithmetic.

Argument 3: Given the right background, any mathematical object can play the role of any position in any mathematical structure. Therefore, NUP-ists have to allow that every mathematical singular term refers (non-uniquely) to every mathematical object. But this *vicious* sort of non-unique reference is surely unacceptable.

In response to argument 3, I simply deny that NUP-ists are committed to this vicious sort of non-unique reference. This goes back to a point I made in section 2: just because all mathematically important facts are structural facts, it does not follow that these are the only facts relevant to the determination of mathematical reference. To take an example, NUP-ists can maintain that we know that '3' does not refer to any function, because it is built into our conception of the natural numbers—*i.e.*, FCNN—that numbers are not functions. More generally, the point here is that the vast majority of ω-sequences fail to satisfy FCNN, and so our numerals do not refer to the objects in those sequences. They only refer to the objects in those ω-sequences that satisfy FCNN. Of course, this is all completely standard; the only non-traditional claim that NUP-ists make is that there may be numerous ω-sequences that satisfy FCNN.

One might object as follows. 'Your argument here shows that we cannot assign to the term "3" any object that is in the extension of the predicate "function". Thus, given that the extension of "function" is *fixed*, it follows that "3" does not refer to any object in this extension. But if we are allowed to reinterpret "function" and "3" at the same time, then things change. Indeed, if we think of interpretations as applying to the language of mathematics as a *whole*, then it seems that "3" could refer to any mathematical object whatsoever. For we could, so to speak, "begin" each interpretation by assigning an object to "3", and then construct the rest of the interpretation around this, being careful that the referent of "3" is in the extension of "number", not in the extension of "function", and so on.'

The author of this objection seems to think of mathematical objects as, so to speak, 'bare particulars'. That is, the objection seems to assume something like

(I) Taken *in themselves*, *i.e.*, without any interpretation present, mathematical objects are all indistinguishable from one another.

Now, I grant that if (I) were true, then NUP-ists would have to admit that '3' could refer to any mathematical object whatsoever. But this would not be a *problem* for NUP, because if (I) were true, then '3' *really could* refer to any mathematical object whatsoever, because all these objects would be indistinguishable from one another. Thus, if (I) were true, then we would have a refutation not of NUP, but of U-platonism! But it seems to me that (I) does not fit very well with platonism anyway. Consider, for instance, $\emptyset$ and $\{\emptyset\}$. Is it in the spirit of platonism to claim that these two objects are indistinguishable and that we just happen to assign the former to '$\emptyset$' and the latter to '$\{\emptyset\}$'? I think it is better (and more in the spirit of platonism) to say that it is part of the *nature* of $\{\emptyset\}$ that it contains $\emptyset$ (or in NUP-ist terms, if x is a referent of '$\{\emptyset\}$', then it is part of x's nature that it contains an object which is a referent of '$\emptyset$'). (Moreover, FBP seems to entail that there *are* mathematical objects with such natures, because it says that all the mathematical objects that possibly could exist actually do exist, and intuitively, it seems that there could exist mathematical objects with such natures.) But if mathematical objects have distinct natures in this way, then it is surely not the case that the term '3'—or more precisely, *our* term '3', *i.e.*, the '3' of FCNN—could refer to any mathematical object at all.

Moving on to argument 2, my response here is to deny that there is a unique sequence of objects such that mathematicians have *that sequence in mind* when they are doing arithmetic, or talking about 'the natural numbers'. Given the platonist thesis that the mathematical realm exists independently of us and our theorizing, we arrive at the result that there may be numerous ω-sequences that satisfy FCNN and only differ from one another in ways that no human being has ever imagined. Moreover, if there *are* numerous ω-sequences that satisfy FCNN, then they are all on a par with respect to our arithmetical thoughts and beliefs; that is, none of them stands out from the others as somehow 'uniquely connected' to our mental states so that we have *it* in mind. This is simply because FCNN is *all* we have in mind when we do arithmetic. FCNN gives us a list of desiderata that need to be satisfied by an ω-sequence in order for it to be a candidate for being the sequence of natural numbers; but if several different sequences satisfy the list of desiderata, then none of them is *the* sequence of natural numbers, because there is nothing *more* that we have in mind, over and above the list of desiderata contained in FCNN, that could settle the matter. Thus, if FCNN does not pick out a unique ω-sequence, then we simply do not have a unique ω-sequence in mind when we do arithmetic.

Now, in *empirical* contexts, when there is no unique object answering to a singular term, we often want to say that something has gone *wrong*—that the singular term suffers a reference failure, or something to that effect. But I have already argued that in mathematical contexts, there is no need to make such a claim. If FCNN does not pick out a unique ω-sequence and, hence, the numerals do not pick out unique objects, it is simply not a problem, because we can accomplish what we want to accomplish in arithmetic—in particular, we can succeed in characterizing the structural facts that we want to characterize—even if FCNN fails to pick out a unique ω-sequence and the numerals fail to denote unique objects. And it is not just that there is no *need* to consider non-unique reference problematic; the fact is that mathematicians *wouldn't* consider it problematic. In other words, is seems to me that the above stance is perfectly consistent with mathematical practice. Now, in saying this, I do not mean to deny that there are a lot of mathematicians who naively think that the numerals have unique referents. What I mean is this: if we pointed out to these mathematicians that it may be that there are numerous ω-sequences that satisfy all of FCNN and only differ from one another in ways that no human being has ever imagined, they would not see this as a problem. Most of them, I think, would say something like this: 'Oh, come on. None of this matters in the least bit, because all of these sequences will serve our needs perfectly well. If they all satisfy FCNN, that's all that matters. There does not have to be a *unique* sequence that satisfies FCNN.'[17]

What about argument 1? Well, I suppose the least misleading way to express my response to this argument is as follows: I deny that the NUP-ist's appeal to non-unique reference brings with it an abandonment of standard semantics. In giving a standard semantics for the language of arithmetic, what we do is assign objects to the singular terms, sets of objects to the one-place predicates, sets of ordered pairs of objects to the two-place predicates, and so on. NUP-ists do *not* want to abandon this practice. They merely want to claim that in using singular terms in arithmetic and providing them with a standard semantics, we make an assumption that is, strictly speaking, false, but nevertheless, very convenient and completely

[17] One might object to the view that I have been developing here by claiming that (U) is already built into FCNN and, hence, that if there is not a unique ω-sequence that satisfies FCNN, then nothing satisfies it. But part of what I am arguing here is that these claims are *wrong*. Now, I suppose there might be some sense in which (U) is built into certain 'untutored' conceptions of the natural numbers, but my claim here is that it is not built into FCNN. That is, it is not built into our 'educated' conception of the natural numbers, *i.e.*, the theorectically infuluenced conception that is inherent in contemporary mathematical practice.

harmless. The assumption is just that there is a unique ω-sequence that satisfies FCNN, or in other words, that numerals have unique referents. The reason this assumption is convenient should be obvious: it is just intuitively pleasing to do arithmetic in this singular-term, standard-semantics way. And the reason the assumption is *harmless* is that we simply are not interested in the differences between the various ω-sequences that satisfy FCNN. In other words, all of these sequences are indistinguishable with respect to the sorts of facts and properties that we are trying to characterize in doing arithmetic, and so no harm can come from proceeding as if there were only *one* sequence here.

One might object as follows. 'I'm wondering what you think the *truth conditions* of, *e.g.*, "3 is prime" are. Of course, you might just say that this sentence is true iff 3 is prime, and you might add that it's "convenient" to take this as being about a particular object, but you don't think that it's *really* about a particular object. On your view, there might be many 3s. Moreover, some of these so-called "3s" might be "4s" in other ω-sequences—indeed, in other ω-sequences that satisfy FCNN. And of course, in the setting of these other ω-sequences, these "3s" will not be prime. So in asking what the truth conditions of "3 is prime" are, what I'm asking is this: According to FBP-NUP, what does the world really need to be *like* in order for it to be the case that "3 is prime" is true?'

Before I say what FBP-NUP-ists take the truth conditions of '3 is prime' to be, let me say that I think there is a mistake in this objection. FBP-NUP-ists admit that there might be many 3s, but they do *not* admit that one of these 3s could be a '4' in another ω-sequence that satisfied FCNN. Now, of course, *all* of these 3s appear in the '4 position' in other ω-sequences, but none of these other ω-sequences satisfies FCNN, because they all have objects in the '4 position' that have the property *being 3*. And we know that these objects have the property *being 3*, because (a) by hypothesis, they appear in the '3 position' in ω-sequences that do satisfy FCNN, and (b) they could not do this without having the property *being 3*, because it is built into FCNN that the number 3 has the property *being 3*.

Given this, I can answer the above question—*i.e.*, 'What does the world need to be like in order for it to be the case that "3 is prime" is true?'—in the obvious way. In order for '3 is prime' to be true, it needs to be the case that there is an object that (a) satisfies all of the desiderata for being 3 and (b) is prime. This, of course, is virtually identical to what traditional U-platonists would say about the truth conditions of '3 is prime'. The only difference is that FBP-NUP-ists allow that it may be that there are *numerous* objects here that make '3 is prime' true.

So it seems to me that none of the three arguments shows that it would be unacceptable to use singular terms in mathematics without there being unique referents for those terms (or that, in point of actual fact, our mathematical singular terms do have unique referents). Therefore, this whole reference-based argument for (U)—or for the superiority of U-platonism over NUP—fails.

The second argument that one might give for favoring U-platonism over NUP, or FBP-NUP, proceeds as follows. 'One of the main reasons for endorsing mathematical platonism, as opposed to some sort of anti-realistic view, is that it delivers the result that undecidable propositions like the continuum hypothesis (CH) have unambiguous truth values. But FBP-NUP doesn't deliver this result, because it maintains that CH is true in some hierarchies of sets and false in others.'

This objection is simply misguided. FBP-NUP is perfectly consistent with the thesis that CH has an unambiguous, or determinate, truth value. It does not *entail* that CH has an unambiguous truth value, but as I will explain, this is not just acceptable, it is *desirable*. What determines whether CH—or any other set-theoretic sentence, for that matter—has an unambiguous truth value is whether it (or its negation) is 'built into our notion of set'. In other words, a set-theoretic sentence has an unambiguous truth value if and only if it has the same truth value in all of the standard models of set theory, where a model of set theory is *standard* if and only if it corresponds to 'our notion of set', or 'what we have in mind in connection with set theory', or 'our full conception of the universe of sets' (FCUS), or something to this effect.[18] Now, in the case of arithmetic, it is plausible to suppose that all sentences have unambiguous truth values, because it is plausible to suppose that FCNN is categorical and, hence, that all of the standard models of arithmetic are isomorphic to one another. But in the case of set theory, this is not so obvious. It *may* be that FCUS is *not* categorical. More specifically, it may be that neither CH nor its negation is built into FCUS, *i.e.*, that CH is true in some standard models and false in others. If this is indeed the case, then CH does not have an unambiguous truth value. But on the other hand, it may be that CH or its negation *is* built into FCUS, and if this is the case, then CH does have an unambiguous truth value.

Thus, FBP-NUP-ists maintain that CH is true in some universes of sets and false in others and that the question of whether CH has an unambiguous truth value is the question of whether it has the same truth value in all of the standard universes. Therefore, FBP-NUP is consistent with the thesis that CH has an unambiguous truth value and also with the thesis

[18] This is slightly oversimplified. Whether CH has an unambiguous truth value depends not just on whether it or its negation is built into FCUS, but also on whether it would be wise to *modify* FCUS in a way that would settle the CH question. (This suggests that there is a relationship of back-and-forth influence between theory and intuition. It is obvious that our theories are influenced by our intuitive pre-theoretic concepts, but our concepts are also influenced by our theorizing. Thus, which models count as 'standard' is influenced by our theorizing as well.)

that CH does *not* have an unambiguous truth value.[19] More generally, FBP-NUP-ists can account for undecidable sentences that have unambiguous truth values *and* undecidable sentences that do not have unambiguous truth values. This is an extremely appealing feature of FBP-NUP. In contrast, traditional U-platonism cannot account for undecidable sentences that do not have unambiguous truth values. If set theory is about a *unique* universe of sets, then CH is either true in that universe or false in it. This, I think, is a serious problem for U-platonism, because one of the dominant opinions about CH among contemporary set theorists—if not *the* dominant opinion—is that CH does not have an unambiguous truth value; thus, if U-platonism cannot account for how this could be so, then that view is implausible. More generally, the problem here is that metaphysical views like platonism should not commit us to answers to questions like 'Does CH have an unambiguous truth value?' Mathematicians should be able to answer such questions *on their own*; they should not have answers *dictated* to them by philosophers of mathematics. Thus, the conclusion to be drawn here is this: far from giving us a reason to favor U-platonism over FBP-NUP, considerations concerning undecidable sentences like CH actually give us reason to favor FBP-NUP over U-platonism, because FBP-NUP enables us to account for *more* here; in particular, it enables us to account for undecidable sentences with unambiguous truth values *and* undecidable sentences without unambiguous truth values.[20]

(This stance on the CH issue gives rise to some epistemological worries

[19] One might argue that FBP-NUP-ists are committed to the thesis that CH has an unambiguous truth value by claiming that (a) we can amalgamate all the universes of sets to form a *single* universe, and (b) FBP-NUP-ists ought to understand CH as being about this particular universe. But the person who denies that CH has an unambiguous truth value—or, more generally, who thinks that FCUS is not categorical—will deny that there is a unique amalgamated universe that clearly contains all the things that legitimately count as sets and none of the things that do not. Moreover, if there *is* a unique amalgamated universe of this sort, then this shows not just that *FBP-NUP-ists* ought to say that CH has an unambiguous truth value, but that *everyone* ought to say this. In other words, it shows that CH does have an unambiguous truth value. The important point to note here is that by *itself*, FBP-NUP is *neutral* with respect to the question of whether there is a unique amalgamated universe that contains all and only things that legitimately count as sets; that is, it is neutral as to whether there is a unique universe that corresponds to our notion of set. Therefore, it is also neutral as to whether all set-theoretic sentences have unambiguous truth values.

[20] These remarks suggest that FBP-NUP-ists can respond to Putnam's [1980] Löwenheim-Skolem argument in essentially the same way that I have responded here to the non-uniqueness argument. In a nutshell, Putnam's stance is that (a) FCNN and FCUS are not categorical and (b) this is a problem for platonism. Now, I actually think that FCNN (if not FCUS) *is* categorical, but in any event, my response to Putnam's argument is simply to take the attitude of this paper one step further and embrace non-categoricity as well as non-uniqueness. In other words, it does not matter whether FCNN or FCUS is cagtegorical, because FBP-NUP is perfectly compatible with the thesis that they are *not* categorical. I discuss this response to Putnam's argument in a bit more detail in chapter IV of my [forthcoming].

about how human beings could know what the various standard models are like, and how they could 'mentally pick out' a particular model as standard. But I think that these worries can be answered.[21])

So it seems to me that the above reference-based argument and this CH-based argument both fail to provide us with a reason to favor U-platonism over NUP, or FBP-NUP. Now, of course, it may be that there is some other argument that U-platonists could use here that is actually cogent, but I do not think this is the case.[22] Therefore, I conclude that there are good reasons for favoring FBP-NUP over U-platonism and no good reasons for favoring U-platonism over FBP-NUP. Moreover, I would also like to add here that, intuitively, FBP-NUP is more in the 'spirit' of platonism than traditional U-platonism is. For in allowing that all the mathematical objects that possibly could exist actually do exist, FBP-NUP-ists eliminate any *arbitrariness* from the mathematical realm. Moreover, when they allow that our mathematical theories might not capture unique parts of the mathematical realm, they are merely acknowledging that the mathematical realm is, in some sense, 'beyond us' and that there may be parts and facets of the mathematical realm that we have never thought about.

One last point. I want to emphasize that this paper does *not* contain an argument in *favor* of platonism. All I have done is defend platonism against a certain attack. I have argued that platonists can avoid this attack if (and only if) they endorse NUP, or FBP-NUP. Thus, in effect, what I have argued is that *if* we decide to endorse platonism, *then* we ought to endorse FBP-NUP. But for all that has been said here, it may be that platonism is false—that there are no such things as abstract objects and, hence, that our mathematical theories are either satisfied only by collections of *concrete* objects (*i.e.*, spatio-temporal objects) or else not satisfied by anything.

[21] Briefly, the reason these worries can be answered is that standard models are not metaphysically special. They are only *sociologically* special, or *psychologically* special. To ask whether some proposition is true in, e.g., the standard model (or class of models) of set theory is just to ask whether it is inherent to *our* notion of set. Thus, since our notion of set is clearly accessible to us, questions about what is true in the standard model (or class of models) of set theory are clearly within our epistemic reach. More generally, the answer to the question 'How do we know what the various standard models of mathematics are like?', is just this: we formulate axioms that are intuitively pleasing (*i.e.*, that jibe with our notion of set, or number, or whatever) and then we prove theorems. This, of course, is exactly true to mathematical practice. And in particular, it is exactly true to the situation regarding CH; what set theorists want—or at any rate, what those set theorists who think that CH has an unambiguous truth value want—is a new axiom that is (a) powerful enough to entail an answer to the CH question and (b) intuitively pleasing.

[22] In chapter III of my [forthcoming], I consider a few other arguments that U-platonists might use here and I show that none of them is cogent.

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ABSTRACT. A response is given here to Benacerraf's (1965) non-uniqueness (or multiple-reductions) objection to mathematical platonism. It is argued that non-uniqueness is simply not a problem for platonism; more specifically, it is argued that platonists can simply embrace non-uniqueness—*i.e.*, that one can endorse the thesis that our mathematical theories truly describe collections of abstract mathematical objects while rejecting the thesis that such theories truly describe unique collections of such objects. I also argue that part of the motivation for this stance is that it dovetails with the correct response to Benacerraf's other objection to platonism, *i.e.*, his (1973) epistemological objection.