**Department of Mathematics**
**California State University Los Angeles**

**Master's Degree Comprehensive Examination in**

**NUMERICAL ANALYSIS**
**SPRING 2023**

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**Instructions:**

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No notes, books, calculators, internet or cell phones are allowed.

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**PART A:** Do only **TWO** problems

1. Given a $3 \times 3$ matrix

$$A = \begin{pmatrix} 2 & 0 & k \\ 0 & 2 & k \\ k & k & 2 \end{pmatrix} \quad \text{with } k \in \mathbb{R}.$$

   (a) Determine *all* values $k$ for which $A$ is

      i. [4 pts] strictly diagonally dominant.

      ii. [4 pts] orthogonal.

      iii. [4 pts] positive definite.

   (b) [7 pts] Find the spectral radius of the Gauss-Seidel iteration matrix for solving $A\mathbf{x} = \mathbf{b}$. Then determine *all* values $k$ for which the Gauss-Seidel iteration converges.

   (c) [6 pts] Perform one step of the Gauss-Seidel iteration for solving $A\mathbf{x} = \mathbf{b}$ with $k = 1$, and $\mathbf{b} = (1, 2, 3)^T$. Use the initial guess $\mathbf{x}_0 = (0, 0, 0)^T$.

2. Given $A \in \mathbb{C}^{n\times n}$. Assume that the eigenvalues of $A$ satisfy $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$.

   (a) [6 pts] Describe the power method to find $\lambda_1$ and its corresponding eigenvector. Does the power method always converge for any initial vector $\mathbf{q}_0$? Explain your answer.

(b) [6 pts] Show that the convergence is linear.

(c) [4 pts] Give one advantage and one disadvantage of the power method, in comparison to the QR method.

(d) [3 pts] Describe briefly how Rayleigh Quotient Iteration improves rate of convergence for power method.

(e) [6 pts] Perform one steps of Rayleigh Quotient Iteration on matrix $\begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix}$ with initial vector $\mathbf{q}_0 = (1, 0)^T$.

3. Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 4 & 0 & 3 \\ 4 & 1 & 2 \\ 8 & 3 & 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 6 \\ 14 \end{pmatrix}.$$

(a) [8 pts] Use Gaussian elimination with partial pivoting to write the matrix $A$ in the form $PA = LU$, where $P$ is a permutation matrix, $L$ is a unit lower triangular matrix and $U$ is an upper triangular matrix.

(b) [6 pts] Use the result from (a) to solve the linear system $A\mathbf{x} = \mathbf{b}$.

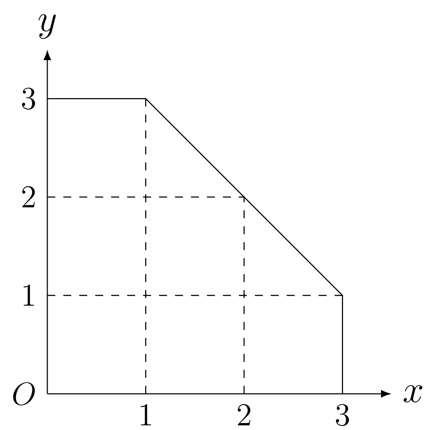
(c) [5 pts] Can the above system be solved using Gaussian elimination without pivoting? Explain your answer.

(d) [3 pts] Given one reason why Gaussian elimination with partial pivoting is preferred in practice over Gaussian elimination without pivoting.

(e) [3 pts] Briefly describe the complete pivoting strategy and its performance in comparison to the partial pivoting.

**PART B:** Do only **TWO** problems

1. Consider the PDE $u_{xx} + 3u_{yy} = 6$ with boundary values $u(x, y) = 2x^2 + y^2$ defined on the region $R$ shown below:




   (a) [5 pts] Find the maximum value of $u$ attained within the region $R$. At what point(s) is this maximum value attained?

   (b) [8 pts] Suppose that we approximate this PDE by the usual 5-point scheme on a square mesh with $\Delta x = \Delta y = h = 1$. Write down the resulting system of linear equations in matrix form $A\mathbf{u} = \mathbf{b}$.

   (c) [5 pts] Derive the local truncation error for the 5-point scheme that you used in part (b).

   (d) [7 pts] Now suppose that the boundary condition along $y = 0$ is replaced by $u_y(x, 0) = 1$ for $0 < x < 3$. Write a second order accurate finite difference equation to approximate the solution at the point $(2, 0)$ in terms of $u(1, 0)$ and $u(2, 1)$.

2. Consider the PDE

$$u_{xx} + xu_{xy} - 2x^2u_{yy} = 0$$
$$u(x, 0) = x^2, \ -\infty < x < \infty$$
$$u_y(x, 0) = 0, \ -\infty < x < \infty.$$

   (a) [3 pts] Determine all values of $x$ for which the given PDE is hyperbolic.

   (b) [4 pts] Determine the slope of the characteristic curves for the PDE at a point $(x, y)$.

   (c) [6 pts] Find the exact values of the coordinates of the point of intersection $R(x_R, y_R)$, where $y_R > 0$, of the characteristic curves through the points $P(2, 0)$ and $Q(4, 0)$.

(d) [3 pts] State the CFL condition that gives the stability condition for the numerical scheme used to approximate the above PDE.

(e) [4 pts] Suppose we approximate the given PDE by a consistent explicit finite difference scheme with $h = k = 1$. Does the scheme converge at the point $R$ you found in part (c)? Explain your reasoning.

(f) [5 pts] Derive a consistent finite difference approximation for the term $xu_{xy}$ that is $O(h, k^2)$. You need not prove that your approximation is consistent.

3. Consider the PDE

$$\begin{aligned} u_t &= u_{xx},\ 0 \le x \le 1,\ t > 0 \\ u(x,0) &= x(1-x),\ 0 \le x \le 1 \\ u(0,t) &= 1,\ u(1,t) = t,\ t > 0. \end{aligned}$$

Suppose we approximate the above PDE by a finite difference scheme

$$-r(1-\theta)u_{i-1,j+1} + (1+2r\theta)u_{i,j+1} - r(1-\theta)u_{i+1,j+1} = r\theta u_{i-1,j} + (1-2r(1-\theta))u_{i,j} + r\theta u_{i+1,j},$$

where $r = k/h^2, k = \Delta t, h = \Delta x$, $u_{i,j}$ approximates $U(ih, jk)$, and $\theta$ is a parameter with $0 \le \theta \le 1$.

(a) [3 pts] Find all value(s) of $\theta$ for which the scheme is implicit.

(b) [10 pts] Let $h = 1/4, k = 1/16$ (that is, $r = 1$) and $\theta = 1/2$. Find the matrices $A, B$ and vector $\mathbf{f}_j$ such that

$$A\mathbf{u}_{j+1} = B\mathbf{u}_j + \mathbf{f}_j,$$

where $\mathbf{u}_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})^T$.

(c) [6 pts] Now let $\theta = 1$. Perform von Neumann analysis to find the value(s) of $r$ for which the scheme is stable.

(d) [3 pts] Give an example of a stable scheme that is consistent with the above PDE but not convergent, if any. If there is none, explain why.

(e) [3 pts] Give an example of a consistent scheme that is second order accurate in both $x$ and $t$, if any. If there is none, explain why.