Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2012

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No calculators.
- Closed books and closed notes.

PART A: Do only **TWO** problems

1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that A has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ with respective eigenvectors $\mathbf{v}_1 = [r \quad -r]^T$ $(r \neq 0)$ and $\mathbf{v}_2 = [s \quad s]^T$ $(s \neq 0)$.

(a) [6 points] Suppose we want to apply the Power Method (with scaling) to find the dominant eigenvalue of this matrix A. If the initial iterate is $\mathbf{x}_0 = [1 \quad 3]^T$, verify the conditions that show the method will converge in this case.

(b) [4 points] With initial iterate $\mathbf{x}_0 = [1 \quad 3]^T$, apply the Power Method with scaling to the given matrix A to find $\mathbf{x}_1$.

(c) [3 points] Give an example of an initial iterate for which the Power Method applied to the given matrix A will *not* converge.

(d) [6 points] Find a QR decomposition of the given matrix A, where Q is an orthogonal matrix and R is an upper triangular matrix.

(e) [3 points] Setting the given matrix $A = A_0$, find the matrix A_1 (the next iterate) in the QR Method for finding the eigenvalues of A.

(f) [3 points] Give one advantage of using the QR Method over the Power Method in finding the eigenvalues of an *arbitrary square symmetric* matrix B.

2. Let

$$B = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}.$$

(a) [8 points] By computing the spectral radius of the iteration matrix, determine whether or not Jacobi iteration converges in solving $B\mathbf{x} = \mathbf{c}$, where $\mathbf{c}$ is an arbitrary 3-vector.

(b) [3 points] Will the Gauss-Seidel iteration converge for the above system? Why or why not?

(c) [6 points] How many iterations are needed to reach an accuracy of 10^{-5} for *each* method respectively?

(d) [8 points] Derive the *general* matrix equation for the SOR iteration (i.e. not for the problem in part (a)). Is the matrix B above a good candidate for the SOR method? Why or why not?

3. (a) [2 points] What is the following decomposition for A called?

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) [6 points] ***Without*** doing any computation, determine the multipliers and pivots used to create this decomposition of A. (Six altogether)

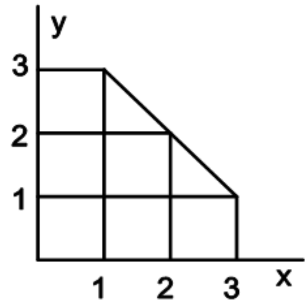
(c) [3 points] Under what condition (on the parameters d_1, d_2, d_3) is the above matrix A nonsingular?

(d) [8 points] Solve the system $A\mathbf{x} = \mathbf{b}$, where A is as given in part (a) and $\mathbf{b} = [0\ 0\ 1]^T$ ***without*** forming A.

(e) [6 points] Prove that the Cholesky decomposition for an arbitrary positive definite matrix A is a special case of its LU decomposition (Hint: Start with $A = LU$ and apply properties of A).

PART B: Do only **TWO** problems

1. (a) Consider the PDE $U_{xx} + U_{yy} = 0$ with boundary values $U(x, y) = x^2 + y^2$ defined on the region R shown below:



   (i) [3 points] What is the maximum value of U attained within the region R. Briefly justify your answer.

   (ii) [8 points] Suppose that we approximate this PDE by the usual 5-point scheme on a square mesh with $h = 1$ (see the figure). Obtain the 3×3 matrix A and 3-vector $\mathbf{b}$ that gives the resulting linear system $A\mathbf{u} = \mathbf{b}$.

   (iii) [3 points] Show that the system $A\mathbf{u} = \mathbf{b}$ found in part (ii) has a unique solution (without solving the system).

   (b) [3 points] Determine all values of k for which the PDE

$$U_{xx} + kU_{xy} + U_{yy} = 0$$

   is elliptic.

   (c) [8 points] By finding the truncation error, determine whether or not the expression

$$\frac{1}{h^2}\big[u(x+h, y+h) - u(x+h, y) - u(x, y+h) + u(x, y)\big]$$

   is a consistent approximation to $u_{xy}(x, y)$.

2. Consider the PDE

$$U_t - 3U_x = 0, \ -\infty < x < \infty, \ t > 0$$

   with initial condition $U(x, 0) = \sin x$.

   (a) [8 points] Solve the above PDE using the method of characteristics.

   (b) [10 points] Give an example of an unstable scheme for the above PDE. Justify that it is unstable for any choice of Δx and Δt.

(c) [7 points] Give an example of an unconditionally stable scheme for the above PDE if there is any. Be sure to justify that it is unconditionally stable. If there is no such scheme, explain why.

3. Consider the PDE

$$U_t - U_{xx} = 0, \;\; 0 \leq x \leq 1, \;\; t \geq 0$$

with boundary and initial conditions

$$U(0,t) = U(1,t) = 0$$

$$U(x,0) = \begin{cases} 2x & 0 \leq x \leq 0.5 \\ 2(1-x) & 0.5 < x \leq 1 \end{cases}$$

Suppose the following scheme is used to approximate the solution to the above PDE

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \frac{1}{2}\Big(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\Big) = 0. \qquad (1)$$

(a) [3 points] Is scheme (1) implicit or explicit? What is the name of the scheme?

(b) [10 points] Show that scheme (1) is second order accurate in both time and space.

(c) [4 points] Give an example of a scheme to approximate the above PDE that is first order accurate in time. You need not prove it.

(d) [8 points] Let $h = \Delta x = 1/4, \; k = \Delta t = 1/16$. Write down the system of equations $A\mathbf{u} = \mathbf{b}$ that results from using scheme (1) to solve the PDE at time $t = 1/16$. Do not solve the system.