Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2016

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No calculators.
- Closed books and closed notes.

PART A: Do only **TWO** problems

1. Let
$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}.$$

   (a) [6 points] Use the definition of a positive definite matrix ($\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x}$) to show that A is positive definite.

   (b) [6 points] Find an upper triangular matrix R with positive diagonal entries so that A can be written as $A = R^T R$.

   (c) [2 points] What is the decomposition in part (b) called?

   (d) [4 points] Suppose an $n \times n$ matrix A has been decomposed into $A = R^T R$, where R is an upper triangular matrix, as in part (b). Explain how you can use this decomposition to solve the system $A\mathbf{x} = \mathbf{b}$.

   (e) [3 points] **Use the result of part (b)** to find $\det(A)$.

   (f) [4 points] Apply Gerschgorin Circle Theorem to determine the lower and upper bounds for all eigenvalues of A.

2. Let

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{pmatrix}.$$

(a) [2 points] Is A strictly diagonally dominant (Yes/No)? Briefly explain why or why not.

(b) [2 points] Is A orthogonal (Yes/No)? Briefly explain why or why not.

(c) [8 points] Show that the Gauss-Seidel iteration scheme for the system $A\mathbf{x} = \mathbf{b}$ converges.

(d) [5 points] Find k such that $\|\mathbf{x}^{(k)} - \mathbf{x}\|_2 \leq 10^{-3}\|\mathbf{x}^{(0)} - \mathbf{x}\|_2$, where $\mathbf{x}^{(k)}$ is the k-th iterate of the Gauss-Seidel iteration.

(e) [8 points] For a given linear system $B\mathbf{x} = \mathbf{d}$, and a splitting $B = M - N$, show that the iterative method $\mathbf{x}^{(k+1)} = P\mathbf{x}^{(k)} + \mathbf{c}$, where $P = M^{-1}N$, is convergent whenever all eigenvalues of P satisfy $|\lambda_i| < 1$ for all i. You may assume that the eigenvectors of P are linearly independent.

3. (a) The 2×2 matrix B has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$, and associated eigenvectors $v_1 = (1,1)^T$ and $v_2 = (1,-1)^T$.

   i. [6 points] If we take the initial vector to be $\mathbf{x}^{(0)} = (2,0)^T$, will the Power Method converge? Justify your answer. What does it converge to? How do you modify the Power Method so that it will converge to the other eigenvector?

   ii. [5 points] Find the matrix B.

   iii. [6 points] Apply one iteration of the QR algorithm to approximate the eigenvalues of B.

   iv. [2 points] Give one advantage and one disadvantage of the Power Method over the QR algorithm.

   (b) [6 points] Show that if an $n \times n$ symmetric matrix A is positive definite ($\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero vectors $\mathbf{x}$), then all eigenvalues of A are positive.

PART B: Do only **TWO** problems

1. Consider the elliptic boundary value problem (BVP):

$$\begin{aligned} &U_{xx} + U_{yy} = 6 \quad \text{for } 0 < x < 3, 0 < y < 3 \\ &U(x,0) = 3x^2, \; U(x,3) = 6x + 3x^2 \quad \text{for } 0 \le x \le 3 \\ &U(0,y) = 0, \; U(3,y) = 6y + 27 \quad \text{for } 0 \le y \le 3 \end{aligned}$$

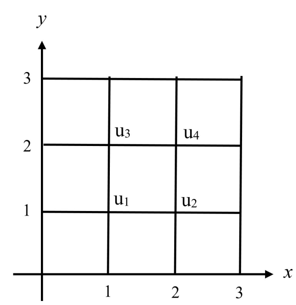
   (a) [4 points] An exact solution of this BVP has the form $U(x,y) = Axy + Bx^2$, where A and B are constants. Find A and B.

   (b) [3 points] Is there more than one correct answer to Part (a)? Briefly explain why or why not.

   (c) [10 points] Use the usual 5-point approximation to $U_{xx} + U_{yy}$ to get a scheme that approximates the given partial differential equation. Then, find the system of 4 linear equations in the 4 unknowns $u_1 = u(1,1), u_2 = u(2,1), u_3 = u(1,2), u_4 = u(2,2)$ that results from applying this scheme to the given BVP on the grid shown below. (Note that $h = 1$).

   (d) [3 points] Explain why the solution to the system of equations in Part (c) is unique.

   (e) [5 points] Now suppose that for $x = 0$ $(0 < y < 3)$, we replace the given boundary values ($U(0,y) = 0$) with the boundary condition $U_x(0,y) = 0$. Approximating U_x by central differences, express the approximate solution $u(0,1)$ in terms of $u(0,2)$ and $u(1,1)$.



2. Given the partial differential equation (PDE)

$$U_t = U_{xx}, \quad (t > 0, 0 < x < 1).$$

Let $u_{i,j}$ be a finite difference approximation to $U(ih, jk) = U(i\Delta x, j\Delta t)$ and let $\lambda = k/h^2$.

(a) Consider the following explicit scheme for approximately solving $U_t = U_{xx}$:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-2,j} - 2u_{i,j} + u_{i+2,j}}{4h^2}.$$

i. [8 points] Show that this scheme is consistent with the given PDE.

ii. [4 points] Given the boundary conditions $U(0, y) = 0, U(1, y) = 0$, take $h = 1/5$ and let $\mathbf{u}_j$ be the 4-vector $\mathbf{u}_j = (u_{1,j}, u_{2,j}, u_{3,j}, u_{4,j})^T$. Find the 4×4 matrix A such that this scheme has the form $\mathbf{u}_{j+1} = A\mathbf{u}_j$.

iii. [3 points] This scheme is *stable* for $\lambda \leq 1/2$. Does this imply that it *converges* for $\lambda \leq 1/2$? Explain briefly why or why not.

(b) [10 points] Now suppose we approximate $U_t = U_{xx}$ by the "tri-level" scheme:

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}.$$

Use the von Neumann (Fourier) method to determine all values of $\lambda = k/h^2$ for which this scheme is stable.

3. (a) [5 points] Determine all values of x for which the following partial differential equation (PDE) is hyperbolic:

$$U_{xx} + 2xU_{xy} - (x^2 - 2)U_{yy} = 0.$$

(b) The hyperbolic PDE

$$U_{xx} - 4U_{yy} = 0$$

has characteristic curves that are straight lines with slopes ± 2. Suppose that the initial data for this PDE is given on the x-axis ($y = 0$) for $-\infty < x < \infty$.

i. [4 points] Sketch the characteristic curves through the point $P(1, 1)$ and indicate where these lines intersect the x-axis. With the help of your sketch, give the domain (interval) of dependence on the x-axis for $U(x, y)$ at P.

ii. [4 points] Write a finite difference scheme that is consistent with $U_{xx} - 4U_{yy} = 0$ on a grid with spacing $\Delta x = \Delta y = 1$ (that is, $h = k = 1$), and solve for $u_{i,j+1}$. (You need not show that your scheme is consistent.)

iii. [4 points] With the help of Part (i) and the Courant-Friedrichs-Levy (CFL) condition, explain why or why not your scheme in Part (ii) converges at the point $P(1, 1)$.

(c) Now consider the first-order hyperbolic initial value problem

$$U_x + 2xU_y = xU \quad \text{for } 0 < x < \infty, y > 0$$
$$U(x, 0) = x^2 \quad \text{for } 0 < x < \infty$$

and let Q be the point with coordinates $(1, 0)$.

i. [4 points] Find the equation of the characteristic curve through Q.

ii. [4 points] Find U as a function of x along the characteristic through Q.