Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2011

Instructions:

- Do exactly **2 problems from Part A AND 2 problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted toward your grade.
- No calculators.
- Closed books and closed notes.

PART A: Do only **2** problems

1. (a) [8 points] Find the LU decomposition of the matrix

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 7 \\ 3 & 11 & 19 \end{pmatrix},$$

that is, find a unit lower-triangular matrix L and an upper-triangular matrix U such that $B = LU$.

(b) Given the linear system $A\mathbf{x} = \mathbf{b}$, where A has been factored as $A = LU$, and

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}.$$

i. [8 points] Solve $A\mathbf{x} = \mathbf{b}$ *without multiplying* the matrices L and U.

ii. [3 points] Give matrices L, D, and U such that $A = LDU$, where L is a unit lower-triangular matrix, D is a diagonal matrix, and U is a unit upper-triangular matrix.

(c) [3 points each] Give one advantage of Gaussian elimination with partial pivoting over each of the following techniques for solving a nonsingular system of n linear equations in n unknowns:

- Gaussian elimination without partial pivoting
- Jacobi iteration

2. (a) [6 points] Let

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{u}^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Perform one Jacobi iteration for solving $A\mathbf{u} = \mathbf{b}$, with starting vector $\mathbf{u}^{(0)}$, to find $\mathbf{u}^{(1)}$.

(b) [7 points] Let B be an arbitrary 4×4 upper-triangular matrix with nonzero diagonal entries. Show that Gauss-Seidel iteration converges for $B\mathbf{x} = \mathbf{b}$, for arbitrary $\mathbf{b}$.

(c) Let

$$C = \begin{pmatrix} 3 & 1 \\ 5 & 7 \end{pmatrix}.$$

C has eigenvalues 2 and 8, and corresponding eigenvectors $[-s, s]^T$ and $[s, 5s]^T$, respectively, where $s \neq 0$.

i. [6 points] Apply two iterations of the Power Method to the matrix C with initial vector $\mathbf{x}^{(0)} = [1, 0]^T$ to obtain $\mathbf{x}^{(2)}$, an approximation to the eigenvector of C corresponding to eigenvalue 8.

ii. [6 points] Explain why the Power Method converges when applied to the matrix C using the initial value $[1, 0]^T$.

3. (a) [7 points] Let A be an $n \times n$ matrix. Show that if $\rho(A) < 1$, then $I - A$ is non-singular (invertible).

(b) [7 points] Let

$$B = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

Use Gershgorin's circle theorem, together with part (a), to show that B is non-singular.

(c) [3 points] Give an example to show that a diagonally-dominant matrix need not be non-singular. Be sure to show that your matrix is non-singular.

(d) [8 points] Let

$$C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Find the matrix resulting from performing one iteration of the QR method (for approximating eigenvalues) on C.

PART B: Do only **2** problems

1. For the initial boundary value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < 1, t > 0\\
&\frac{\partial u}{\partial x} = u + 1,\ x = 0, t > 0\\
&\frac{\partial u}{\partial x} = -u,\ x = 1, t > 0\\
&u(x,0) = x,\ 0 \le x \le 1
\end{aligned}$$

   (a) [10 points] Show that the scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

   is consistent with the differential equation.

   (b) [15 points] Let $h = 1/10$ and $k = 1/200$. Using central differencing for the boundary data and the scheme above, compute $u_{0,2}$ and $u_{10,2}$.

2. (a) [7 points] By finding the truncation error in approximating the ordinary derivative $f''(x)$ by

$$D_h(f) = \frac{f(x+2h) - 2f(x) + f(x-2h)}{h^2}$$

   determine whether or not this is a *consistent* approximation to $f''(x)$.

   (b) [2 points each] Consider the parabolic PDE $u_t = u_{xx}$.

   i. Explain what it means for a difference scheme for this PDE to be stable.

   ii. Can a consistent explicit scheme approximating this PDE be stable but not convergent? Explain why or why not?

   iii. Give an example of an explicit scheme approximating this PDE which is *not* stable. (You need not prove that your scheme is not stable.)

iv. Explain why the concept of stability does not apply to a difference scheme that is approximating an elliptic PDE.

(c) Consider the PDE $x\dfrac{\partial^2 u}{\partial y \partial x} - \dfrac{\partial^2 u}{\partial^2 y} = 0.$

i. [3 points] Determine the values of (x, y) for which this PDE is hyperbolic.

ii. [4 points] Determine the characteristic curves for this PDE.

iii. [3 points] Can an elliptic PDE generate characteristic curves? Why or why not?

3. (a) Consider the finite difference scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} + \frac{u_{i+1,j} - u_{i,j}}{h} = 0$$

for the PDE $u_t + u_x = 0$.

i. [6 points] Show that the scheme is unstable.

ii. [3 points] Modify the scheme to make it stable.

iii. [4 points] Write down an explicit consistent scheme for $u_t + u_x = 0$ that is unconditionally stable if there is any. If there is none, explain why.

(b) Consider the PDE

$$x^2 u \frac{\partial u}{\partial x} + e^{-y} \frac{\partial u}{\partial y} = -u^2$$
$$u(x, 0) = 1, \;\; 0 < x < \infty$$

i. [6 points] Find the equation for the characteristic curve through the point $(s, 0)$.

ii. [6 points] Find the equation for the characteristic curve through the point $(1, 1)$ and the value of the exact solution of the given IVP at $(1, 1)$.