Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
Spring 2013

Do EXACTLY 2 problems from part I AND EXACTLY two problems from part II. If you do more than two problems in either part of the exam, ONLY THE FIRST TWO will be graded.

NO notes or books; NO graphing Calculators

Part I (Do EXACTLY two problems)

1. Let $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 8 & 0 \\ -2 & 0 & 24 \end{pmatrix}$.

   a. [7%] Find the LU factorization of A, where L is a unit lower triangular and U is an upper triangular matrix.

   b. [5%] Find the LDL^T factorization of A, where L is a unit lower triangular and D is a diagonal matrix.

   c. [5%] Find the $R^T R$ factorization of A, where R is an upper triangular matrix with positive diagonal entries. What is this factorization called?

   d. [8%] Show that a symmetric matrix A is positive definite if and only if it can be factored into $R^T R$, where R is an upper triangular matrix with positive diagonal entries.

2. a. [6%] Let A be an n x n matrix. Give three conditions involving the matrix A that together guarantee that the Power Method (with scaling) will converge to the largest modulus eigenvalue of A.

   b. [6%] Give an example of a 2 x 2 matrix B (and an initial vector $\mathbf{x}^{(0)}$) that violates one of the conditions you gave in part **a**, but for which the Power Method converges. Show that your example does violate the stated condition and show that convergence does occur for your example.

   c. [10%] Find the QR decomposition (Q orthogonal and R upper triangular) of the matrix $C = \begin{pmatrix} 1 & 3 \\ 4 & 7 \end{pmatrix}$.

   d. [3%] Suppose we want to find the eigenvalues of the matrix C in part **c** by the QR method. Letting $C_0 = C$, use the result of part **c** to find the first iterate, C_1, for this method.

3. a. [6%] Let T be an arbitrary nonsingular 3 x 3 upper-triangular matrix. By finding the spectral radius of the Gauss-Seidel iteration matrix for T. prove that this method applied to the linear system T**x** = **b** (**b** arbitrary) always converges.

b. [7%] Let A be an arbitrary strictly diagonally dominant 3 x 3 matrix. Show that the spectral radius of the Jacobi iteration matrix for A is less than 1 (and thus prove that Jacobi iteration applied to the linear system A**x** = **b** (**b** arbitrary) always converges).

c. [6%] Let E be an arbitrary 3 x 3 matrix with spectral radius less than 1. Prove that (I - E) is nonsingular, where I is the 3 × 3 identity matrix.

d. [6%] In solving a linear system C**x** = **b**, where C is an arbitrary nonsingular n x n matrix and **b** is an arbitrary *n*-vector, give one advantage and one disadvantage of a direct method (such as Gaussian elimination) compared to an iterative method (such as Jacobi iteration).

Part II (Do EXACTLY two problems)

1. Consider the homogeneous PDE

$$u_t + 2u_x = 0, \qquad u(0,x) = x^2 + 1, \qquad t > 0, x > 0$$

where $h = \Delta x, k = \Delta t$, and $u_{i,j} \approx u(ih, jk)$.

a. [2%] State the CFL condition for a first order equation under which a numerical scheme is stable.

b. [6%] Write down a numerical scheme that uses first order differencing in both time and space to approximate the solution to the above PDE. What is the CFL condition for the scheme you wrote?

c. [7%] Show that the scheme you found in part **b** is consistent with the given PDE. What is the order of accuracy?

d. [4%] Calculate the exact value of u_Q where Q is a point that lies on the characteristic curve through $P(0,3)$.

e. [6%] Find the solution to the above PDE by using the method of characteristic.

2. Consider the Boundary Value Problem (BVP) below:

$$9u_{xx} + u_{yy} = 0, \quad 0 < x < 1 \text{ and } 0 < y < 1; \qquad u(0,y) = -9y^2,\ u(1,y) = 1 - 9y^2, \quad 0 \le y \le 1$$
$$u(x,0) = x^2, \quad u(x,1) = x^2 - 9, \quad 0 \le x \le 1$$

a. [5%] Using the central difference to approximate the derivatives, write a finite difference scheme for the above BVP and simplify it.

b. [7%] Derive the local truncation error for the scheme you found in part **a**. Is this scheme consistent with the PDE? Explain your answer.

c. [3%] The PDE is elliptic in what region? Explain.

d. [7%] Draw the given domain for the BVP and use $h = k = 1/3$ to approximate u at the resulting four interior mesh points. Simplify your answer into a linear system Au = b (Do not solve).

e. [3%] Will the system Au = b have unique solution? Explain *without* doing any calculation. (You can assume max-min principle for the PDE.)

3. Consider the following explicit "5-point" scheme for approximately solving the partial differential equation (PDE) $U_t = U_{xx}$:

$$[u_{i,j+1} - u_{i,j}] / k = [u_{i-2,j} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i+2,j}] / 5h^2$$

a. [10%] Show that this scheme is consistent with the PDE $U_t = U_{xx}$.

b. [3%] Letting $r = k/h^2$, solve this scheme for $u_{i,j+1}$.

c. [4%] Suppose we want to use the given scheme to solve $U_t = U_{xx}$ in the region $0 < x < 1$, $t > 0$ with boundary conditions $U(0, t) = 0$, $U(1, t) = 0$ for all t. If $h = 1/N$, find the matrix A (of order N - 1) so that the scheme in part ***b*** takes the form $\mathbf{u}_{j+1} = A\mathbf{u}_j$ (where $\mathbf{u}_j$ is the vector with components $u_{1,j}, u_{2,j}, \ldots, u_{N-1,j}$)

d. [5%] Show that for $r < 5/4$ the spectral radius of the matrix A of part **c** is less than 1 (and thus the given scheme is stable for $r < 5/4$).

e. [3%] Use the result of parts **a** and **d** to prove that this scheme *converges* for $r < 5/4$.