**Department of Mathematics**
**California State University Los Angeles**

**Master's Degree Comprehensive Examination in**

**NUMERICAL ANALYSIS**
**SPRING 2018**

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**Instructions:**

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No notes, books, calculators, or cell phones may be used during this exam.

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**PART A:** Do only **TWO** problems

1. (a) [4 points] Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times p$ matrix. Determine the number of *multiplications* (of real numbers) needed to compute the product $AB$. **Show your work**.

   (b) Let
$$C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix}.$$

   i. [8 points] Find a permutation matrix $P$, a unit lower triangular matrix $L$, and an upper triangular matrix $U$ such that $PC = LU$.

   ii. [2 points] To solve $C\mathbf{x} = \mathbf{b}$ (where $\mathbf{b}$ is an arbitrary 3-vector) by Gaussian elimination with partial pivoting, which operation should be performed first?

   (c) [8 points] Suppose that the LU decomposition of a matrix $M$ is given by
$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and that } \mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

   Use this LU decomposition of $M$ with the forward/backward substitution method to solve $M\mathbf{x} = \mathbf{c}$.

   (d) [3 points] Give one advantage of the Gaussian elimination method over the Gauss-Seidel iterative method in solving a system of linear equations.

2. (a) Let $A = \begin{pmatrix} 1 & k & k \\ k & 1 & 0 \\ k & 0 & 1 \end{pmatrix}$, where $k$ is a real number.

Determine all values of $k$ for which

i. [3 points] $A$ is strictly diagonally dominant.

ii. [4 points] $A$ is positive definite.

iii. [3 points] $A$ is orthogonal.

iv. [6 points] The Jacobi iteration method applied to $A\mathbf{x} = \mathbf{b}$ (where $\mathbf{b}$ is an arbitrary 3-vector) converges.

(b) [6 points] Let $B = \begin{pmatrix} 1 & k & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$, where $k$ is a real number. Determine the values of $k$ for which the Gauss-Seidel iteration method applied to $B\mathbf{x} = \mathbf{b}$ (where $\mathbf{b}$ is an arbitrary 3-vector) converges.

(c) [3 points] Give one advantage of the Gauss-Seidel method over the Gaussian elimination method in solving a system of linear equations.

3. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$. Let $\mathbf{q}_0 = (-1, 3)^T$.

(a) [4 points] Find the associated eigenvectors of $A$.

(b) [6 points] Will the Power Method converge when applied to the matrix $A$ with starting vector $\mathbf{q}_0$? If yes, which eigenpair of $A$ will it converge to? If no, explain why. Justify your answer without performing any Power Method iterations.

(c) [7 points] Let $\lambda$ be an eigenvalue of an $n \times n$ matrix $B$ with associated eigenvector $\mathbf{v}$ and let $\rho$ be a real number with $\rho \neq \lambda$. Show that $1/(\lambda - \rho)$ is an eigenvalue of the matrix $(B - \rho I)^{-1}$ with associated eigenvector $\mathbf{v}$.

(d) [8 points] Perform one iteration of the *inverse* Power Method *with shift* $\rho = 4$. Which eigenvector of $A$ will this method converge to? What is the ratio of convergence?

**PART B:** Do only **TWO** problems

1. (a) [6 points] Given the partial differential equation (PDE)

$$xu_{xx} - yu_{yy} = \sin x,$$

determine regions in the $xy$-plane (values of $(x, y)$) such that this PDE is:

   i. elliptic
   ii. parabolic
   iii. hyperbolic

   (b) For the initial-boundary value problem

$$\begin{aligned} &u_t = 4u_{xx} \quad \text{for } 0 \le x \le 1, t > 0 \\ &u(x,0) = x(1-x) \quad \text{for } 0 \le x \le 1 \\ &u(0,t) = 0, \;\; u(1,t) = 0 \quad \text{for } t > 0 \end{aligned}$$

   i. [9 points] Explain (in one sentence each) what it means for a finite difference approximation to the given initial-boundary value problem to be each of the following: consistent; stable; convergent.
   ii. [4 points] Construct a finite difference approximation to the given initial-boundary value problem that is consistent and stable (for the values of $k/h^2$ that you specify). You need *not show* that your scheme is consistent or stable.
   iii. [3 points] Is it possible for a finite difference approximation to the given initial-boundary value problem to be consistent but not stable? If so, give an example; if not, explain why not.
   iv. [3 points] Is it possible for a consistent finite difference approximation to the given initial-boundary value problem to be stable but not convergent? If so, give an example; if not, explain why not.

2. Suppose that the function $u(x, y)$ defined on a square $[0, 3] \times [0, 3]$ satisfies

$$yu_{xx} + xu_{yy} = 0 \tag{1}$$

with boundary conditions:

$$\begin{aligned} &u(x,0) = x, \;\; u(x,3) = 6 \\ &u(0,y) = 2y, \;\; u(3,y) = 3 + y \end{aligned}$$

   (a) [4 points] What are the maximum and minimum values achieved by $u(x, y)$ to the above boundary value problem? At what points $(x, y)$ do they occur?
   (b) [10 points] Find the system of linear equations that results from solving this boundary value problem using the usual 5-point scheme with $\Delta x = \Delta y = 1$. Write your system in the form $A\mathbf{u} = \mathbf{b}$.

(c) [4 points] Explain why the solution to the system of equations in part (b) exists and is unique.

(d) [7 points] Now suppose that the boundary condition $u(x,3) = 6$ is replaced by $u_y(x,3) = 2x$. Write a second order accurate finite difference equation to approximate the solution at the point $(2,3)$ in terms of $u(1,3)$ and $u(2,2)$.

3. (a) [6 points] The one-way wave equation

$$u_t - 3u_x = 0,\ \ u(x,0) = f(x),\ \ t > 0, x > 0 \tag{2}$$

could be solved using the scheme

$$\begin{aligned} &\frac{u_{i,j+1} - u_{i,j}}{k} - 3\frac{u_{i+1,j} - u_{i,j}}{h} = 0 \\ &u_{i,0} = f(ih), \end{aligned}$$

where $h = \Delta x,\ k = \Delta t$ and $u_{i,j} \approx u(ih, jk)$. Determine the condition for stability for the above scheme.

(b) Suppose

$$\begin{aligned} &u_x - 3xu_y = x,\ \ y > 0, -\infty < x < \infty \\ &u(x,0) = 2x, -\infty < x < \infty. \end{aligned}$$

i. [7 points] Calculate the value of $y$ so that $Q(1,y)$ is on the characteristic curve through $P(2,0)$. Sketch the characteristic curve and label the points $P$ and $Q$.

ii. [6 points] Compute the exact value of $u_Q$, where $Q$ is the point found in (i).

iii. [6 points] Use the method of numerical characteristics to calculate the first approximations to the value of $y_Q$ and $u_Q$.