DIRECT SUM CANCELLATION OF NOETHERIAN MODULES

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Abstract. Let A, B and C be modules over a unital ring R such that C is Noetherian and $A \oplus C \cong B \oplus C$. Even though A and B need not be isomorphic, we show that they have isomorphic submodule series, and, equivalently, that A and B are indistinguishable by functions on the category of R-modules that respect short exact sequences.

1. Introduction

Throughout this paper, R will be a fixed unital ring, R-**Mod**, the category of left R-modules and R-**Noeth**, the full subcategory of all Noetherian left R-modules.

An old and important question of module theory is the following:

Suppose we have modules $A, B, C \in R$-**Mod** such that $A \oplus C \cong B \oplus C$. What can be said about the relationship between A and B? In particular, is $A \cong B$?

In the most general case, A and B could be quite different. For example, if $A = C$ is a free R module with an infinite basis, and $B = 0$, then $A \oplus C \cong B \oplus C$, even though A and B are completely unrelated.

Thus we are led to consider various finiteness conditions on the modules. For example, suppose A, B and C are Noetherian modules. It is well known that, in this situation, $A \oplus C \cong B \oplus C$ does not imply that $A \cong B$. One standard example of this, due to Kaplansky and Swan [9] is the following:

Example 1.1. *Let $R = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$, the coordinate ring of the unit sphere. We will write x, y, z for the images of X, Y, Z in R. Let $\eta\colon R \oplus R \oplus R \to R$ be the R-module homomorphism defined by $\eta(a, b, c) = ax + by + cz$. Since $\eta(x, y, z) = 1$, this homomorphism is surjective. Let $P = \ker \eta$, then we get the short exact sequence*

$$0 \to P \to R \oplus R \oplus R \xrightarrow{\eta} R \to 0.$$

Since R is projective, this sequence splits to give $(R \oplus R) \oplus R \cong P \oplus R$. In [9, Theorem 3] *and* [8, 11.2.3] *a topological argument is used to show that $P \not\cong R \oplus R$.*

In spite of this failure of cancellation up to isomorphism, we will show in the main theorem of this paper, 5.5, that if $A \oplus C \cong B \oplus C$ with $C \in R$-**Noeth**, then A and B have isomorphic submodule series. That is, there are submodule series $0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$ and $0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$ and a permutation of the indices σ, such that $A_i/A_{i-1} \cong B_{\sigma(i)}/B_{\sigma(i)-1}$ for $i = 1, 2, \ldots, n$.

We show that this is true for the example above:

Example 1.1 (continued)**.** *It is easily checked that the homomorphism $\tau\colon R \oplus R \oplus R \to P$ given by $\tau(a, b, c) = (a, b, c) - \eta(a, b, c)(x, y, z)$ is the projection from $R \oplus R \oplus R$ onto P. Thus P is generated by $\tau(1, 0, 0)$, $\tau(0, 1, 0)$ and $\tau(0, 0, 1)$.*

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Note that $\tau(x, y, z) = 0$ *and, more generally,* $\tau(a, b, c) = 0$ *if and only if* (a, b, c) *is a multiple of* (x, y, z)*. With this fact, a simple calculation shows that* τ *is monic when restricted to* $R\oplus R\oplus 0$*, so that the submodule* $Q = \tau(R\oplus R\oplus 0) = R\tau(1, 0, 0) + R\tau(0, 1, 0)$ *is isomorphic to* $R \oplus R$*.*

To investigate the quotient module P/Q *we define the homomorphism* $\gamma\colon R \to P/Q$ *by* $\gamma(c) = \tau(0, 0, c)+Q$*. This homomorphism is surjective by construction and a calculation shows that* $\ker\gamma = Rz$*. Thus* $P/Q \cong R/Rz$*.*

Since R *is a domain, we also have* $Rz \cong R$*. Thus* $0 \le Q \le P$ *and* $0 \le R\oplus Rz \le R \oplus R$ *are isomorphic submodule series for* P *and* $R \oplus R$ *with factors isomorphic to* $R \oplus R$ *and* R/Rz*.*

The natural way to prove the theorems of this paper is to record the information we need about the categories R-**Mod** and R-**Noeth** in a monoid, to be called $M(R$-**Mod**$)$, and then use theorems about monoids to prove cancellation rules for modules. Thus, in Section 2 we define strongly separative, refinement and Artinian monoids, and show in Theorem 2.9 how they are related. In Section 3 we construct the monoid $M(R$-**Mod**$)$ with submonoid $M(R$-**Noeth**$)$, and show its universal property. In Section 4 we define a monoid homomorphism Klen from $M(R$-**Noeth**$)$ to an Artinian monoid which is needed to apply Theorem 2.9 to $M(R$-**Noeth**$)$. Finally, in Section 5 we prove that $M(R$-**Noeth**$)$ is strongly separative and show some important consequences of this fact for module cancellation questions.

The results of this paper form a part of the author's Ph.D. thesis [3]. The author wishes to thank Pere Ara for valuable suggestions made at the draft stage of this paper, and Ken Goodearl for help at all stages in its evolution.

2. Strongly Separative, Refinement and Artinian Monoids

The monoids to be constructed in this paper arise from categories of modules. Since the objects of such a category do not, in general, form a set, we must allow the possibility that the elements of a monoid might form a proper class. Thus we will consider a monoid to be a <u>class</u> with an associative operation and an identity element.

All monoids in this paper will be commutative, so we will write $+$ for the monoid operation and 0 for the identity element of all monoids, unless this conflicts with an existing usage. We write $\{0, \infty\}$ for the two element monoid such that $\infty+\infty = \infty$.

Every monoid M has a (translation invariant) preorder defined by

$$a \le b \iff \exists\, c \in M \text{ such that } a + c = b.$$

We will need to distinguish certain submonoids of M which behave well with respect to this order:

Proposition 2.1. *For a nonempty subclass* I *of a monoid* M*, the following are equivalent:*

(1) $(\forall a, b \in M)$ $(a, b \in I \iff a + b \in I)$
(2) I *is a submonoid of* M *and* $(\forall a, b \in M)$ $(a \le b \in I \implies a \in I)$
(3) $I = \gamma^{-1}(0)$ *for some monoid homomorphism* $\gamma\colon M \to \{0, \infty\}$*.*

Proof. Easy. □

Definition 2.2. *A nonempty subclass* I *of a monoid* M *satisfying any of the conditions of this proposition is called an* **order ideal** *of* M*.*

An order ideal, $I \subseteq M$, is a subclass of a monoid which not only preserves the monoid operation, but also the order. More precisely, if $a, b \in I$ then $a \leq b$ with respect to the preorder defined in I if and only if $a \leq b$ with respect to the preorder defined in M.

In the semigroup literature, an **ideal** of a commutative semigroup S is defined to be a subset $J \subseteq S$ such that $a \geq b \in J$ implies $a \in J$. Such a subset of a commutative monoid M would be a subsemigroup but not, in general, a submonoid of M.

Proposition 2.3. *For a monoid M, the following are equivalent:*

(1) $(\forall a, b, c \in M)$ $(a + c = b + c$ and $c \leq a \implies a = b)$
(2) $(\forall a, b \in M)$ $(2a = a + b \implies a = b)$
(3) $(\forall a, b, c \in M)$ $(a + 2c = b + c \implies a + c = b)$
(4) $(\forall a, b, c \in M)(\forall n \in \mathbb{N})$ $(a + (n + 1)c = b + nc \implies a + c = b)$
(5) $(\forall a, b \in M)(\forall n \in \mathbb{N})$ $((n + 1)a = na + b \implies a = b)$
(6) $(\forall a, b, c \in M)(\forall n \in \mathbb{N})$ $(a + c = b + c$ and $c \leq n(a + b) \implies a = b)$

Proof. The equivalence of 1–5 is easy to prove, and 1 is an easy consequence of 6, so we prove here only that 4 and 5 imply 6 ...

Suppose $a + c = b + c$ with $c \leq n(a + b)$ for some $n \leq \mathbb{N}$. Then there is some c' such that $c + c' = n(a + b)$. Adding c' to the equation $a + c = b + c$ gives $(n + 1)a + nb = na + (n + 1)b$. Using 4, we can cancel nb from both sides of this equation to get $(n + 1)a = na + b$. Then, from 5, we get $a = b$. ☐

Definition 2.4. *A monoid M is* **strongly separative** [2] *if it satisfies any of the conditions of the preceding proposition.*

In the remainder of this section, we will use 1 of Proposition 2.3 as our definition of strong separativity.

The second monoid property that we will need to discuss is refinement:

Definition 2.5. *A monoid M has* **refinement** [10] [4] [12] *if for all $a_1, a_2, b_1, b_2 \in M$ with $a_1 + a_2 = b_1 + b_2$, there exist $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that*

$$a_1 = c_{11} + c_{12} \qquad a_2 = c_{21} + c_{22}$$
$$b_1 = c_{11} + c_{21} \qquad b_2 = c_{12} + c_{22}.$$

It is convenient to record refinements using matrices. The refinement of $a_1 + a_2 = b_1 + b_2$ from the definition would be written

$$\begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \end{array}$$

This means that the sum of the entries in each row (column) equals the entry labeling the row (column).

For the proof of the next two lemmas, we note that a refinement monoid M has the following easily proved decomposition property: If $a, b_1, b_2, \ldots, b_n \in M$ with $a \leq b_1 + b_2 + \ldots + b_n$, then there are $a_1, a_2, \ldots, a_n \in M$ such that $a = a_1 + a_2 + \ldots + a_n$ and $a_i \leq b_i$, for $i = 1, 2, \ldots, n$.

Lemma 2.6. *Let M be a refinement monoid, $a_0, b_0, c_0 \in M$ such that $a_0 + c_0 = b_0 + c_0$ with $c_0 \le a_0$. Then there is a refinement matrix*

$$\begin{array}{c c} & \begin{array}{cc} b_0 & c_0 \end{array} \\ \begin{array}{c} a_0 \\ c_0 \end{array} & \begin{pmatrix} d_1 & a_1 \\ b_1 & c_1 \end{pmatrix} \end{array}$$

such that $c_1 \le a_1$.

Proof. This proof is extracted from [2, Lemma 2.7] where it is used to show a related result.

From $a_0 + c_0 = b_0 + c_0$ we get a refinement matrix of the form

$$\begin{array}{c c} & \begin{array}{cc} b_0 & c_0 \end{array} \\ \begin{array}{c} a_0 \\ c_0 \end{array} & \begin{pmatrix} d' & a' \\ b' & c' \end{pmatrix} \end{array}.$$

Since $c' \le c_0 \le a_0 = d' + a'$, we can write $c' = d'' + c_1$ where $d'' \le d'$ and $c_1 \le a'$. Since $d'' \le d'$, we can write $d' = d'' + d_1$. Setting $a_1 = d'' + a'$ and $b_1 = d'' + b'$ gives the required refinement matrix. Further, we have $c_1 \le a' \le a' + d'' = a_1$. □

In a refinement monoid, we can get cancellation results for $a + c = b + c$ similar to those of 2.3 even if the whole monoid is not strongly separative. What is needed is that c lies in a strongly separative order ideal:

Lemma 2.7. *Let I be a strongly separative order ideal in a refinement monoid M. Then*

(1) $(\forall a, b \in M)(\forall c \in I)$ $(a + c = b + c \text{ and } c \le a \implies a = b)$
(2) $(\forall a, b \in M)(\forall c \in I)(\forall n \in \mathbb{N})$ $(a + (n+1)c = b + nc \implies a + c = b)$
(3) $(\forall a, b \in M)(\forall c \in I)(\forall n \in \mathbb{N})$ $(a + c = b + c \text{ and } c \le n(a + b) \implies a = b)$

Proof.

(1) Since $c \le a$ we can use 2.6 to get the refinement matrix

$$\begin{array}{c c} & \begin{array}{cc} b & c \end{array} \\ \begin{array}{c} a \\ c \end{array} & \begin{pmatrix} d_1 & a_1 \\ b_1 & c_1 \end{pmatrix} \end{array}$$

with $c_1 \le a_1$. We have $a_1, b_1, c_1 \le c$, so $a_1, b_1, c_1 \in I$. Since I is strongly separative, the equation $a_1 + c_1 = b_1 + c_1$ implies $a_1 = b_1$. Thus $a = d_1 + a_1 = d_1 + b_1 = b$.

(2) We have $(a + c) + nc = b + nc$ with $c \le a + c$, so using 1, we can cancel c repeatedly to get $a + c = b$.

(3) Since $c \le na + nb$, we can write $c = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ where $a_i \le a$ and $b_i \le b$ for $i = 1, 2, \ldots, n$. Thus

$$a + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = b + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

For each $i = 1, 2, \ldots, n$ we have $a_i, b_i \le c$, so a_i and b_i are in I. Using 1, these elements can be canceled from the above equation one by one to leave $a = b$.

□

Notice in 2.6 that we started with the hypothesis of 2.3.1 and produced similar relationships $a_1 + c_1 = b_1 + c_1$ with $c_1 \leq a_1$, for elements a_1, b_1 and c_1 such that $a_1 \leq a_0$, $b_1 \leq b_0$ and $c_1 \leq c_0$. Repeated application of the lemma then gives a chain of such relationships, $a_n + c_n = b_n + c_n$ with $c_n \leq a_n$, for $n = 1, 2, \ldots$, such that $a_0 \geq a_1 \geq a_2 \geq \ldots$, $b_0 \geq b_1 \geq b_2 \geq \ldots$ and $c_0 \geq c_1 \geq c_2 \geq \ldots$. If we can limit these descending chains by some chain condition then we can get strong separativity in the monoid.

Since a monoid M has, in general, a preorder rather than a partial order, it could have distinct elements a, b such that $a \leq b \leq a$. This would allow the "strictly decreasing" sequence $a \geq b \geq a \geq b \geq \ldots$. To avoid this problem we will define our descending chain condition as follows:

Definition 2.8. *Let M be a monoid.*

(1) *If $X \subseteq M$ is a subclass, then $a \in X$ is* **minimal in** *X if for all $b \in X$, $b \leq a$ implies $a \leq b$.*
(2) *M is* **Artinian** *if every nonempty subclass of M has a minimal element.*

If the monoid M happens to be partially ordered by the relation $\leq$, then these definitions coincide with the usual ones for partially ordered classes.

Two simple examples of Artinian monoids are $(\mathbb{N}, +)$ and $(\mathbf{Ord}, \max)$, where **Ord** is the class of ordinal numbers.

Theorem 2.9. *Let M be a refinement monoid and K an Artinian monoid. If there is a monoid homomorphism $\sigma\colon M \to K$ such that $\sigma(2a) \leq \sigma(a)$ implies $a = 0$ for any $a \in M$, then M is strongly separative.*

Proof. Suppose $a, b, c \in M$ such that $a + c = b + c$ and $c \leq a$. We will show that $a = b$...

Define

$$\mathcal{T} = \{(a', b', c', d') \in M^4 \mid a' + c' = b' + c',\ a = d' + a',\ b = d' + b' \text{ and } c' \leq a'\}.$$

Let $\mathcal{C} \subseteq M$ be the projection of $\mathcal{T}$ onto the third component. $\mathcal{C}$ is not empty since $(a, b, c, 0) \in \mathcal{T}$. Let $c_0 \in \mathcal{C}$ be chosen such that $\sigma(c_0)$ is minimal in $\sigma(\mathcal{C})$, and let a_0, b_0, d_0 be such that $(a_0, b_0, c_0, d_0) \in \mathcal{T}$.

From Lemma 2.6, there is a refinement of $a_0 + c_0 = b_0 + c_0$,

$$\begin{array}{cc} & \begin{array}{cc} b_0 & c_0 \end{array} \\ \begin{array}{c} a_0 \\ c_0 \end{array} & \begin{pmatrix} d_1 & a_1 \\ b_1 & c_1 \end{pmatrix} \end{array}$$

such that $c_1 \leq a_1$. Thus $a_1 + c_1 = b_1 + c_1$, $a = (d_0 + d_1) + a_1$, $b = (d_0 + d_1) + b_1$, that is, $(a_1, b_1, c_1, d_0 + d_1) \in \mathcal{T}$ and $c_1 \in \mathcal{C}$. Since $c_1 \leq c_0$, we have $\sigma(c_1) \leq \sigma(c_0)$, and then the minimality of $\sigma(c_0)$ implies $\sigma(c_0) \leq \sigma(c_1)$.

From $c_1 \leq a_1$, we get $2\sigma(c_0) \leq 2\sigma(c_1) \leq \sigma(c_1) + \sigma(a_1) = \sigma(c_1 + a_1) = \sigma(c_0)$. By our hypotheses, this implies $c_0 = 0$. Thus $a_0 = b_0$ and $a = d_0 + a_0 = d_0 + b_0 = b$. $\square$

To prove the main theorem of this paper, we will use Theorem 2.9, but it is nonetheless worthwhile to note the special case when M and K coincide:

Corollary 2.10. *If M is an Artinian refinement monoid such that for all $a \in M$, $2a \leq a$ implies $a = 0$, then M is strongly separative.*

For other cancellation properties of Artinian refinement monoids, see [3].

3. Monoids from Modules

The purpose of the current section is to construct monoids which will encode the properties of certain subcategories of R-**Mod** with respect to short exact sequences:

Definition 3.1. *A* **Serre subcategory of** R-**Mod***, is a full subcategory* $\mathcal{S}$ *of* R-**Mod** *such that for every short exact sequence* $0 \to A \to B \to C \to 0$ *in* R-**Mod***,* $B \in \mathcal{S}$ *if and only if* $A, C \in \mathcal{S}$.

In particular, a Serre subcategory is closed under taking submodules, factor modules and finite direct sums. The zero module is an object in any Serre subcategory. The only Serre subcategories we will need for this paper are R-**Noeth** and R-**Mod**.

Though we are using here the nomenclature and notation of category theory, we will only be interested in full subcategories of R-**Mod**. So we will think of categories as subclasses of the objects of R-**Mod**, and modules as elements, rather than objects, of these categories.

For each Serre subcategory $\mathcal{S}$ of R-**Mod** we will construct a monoid $M(\mathcal{S})$ whose elements are equivalence classes of modules:

Definition 3.2. *Let* $A, B \in R$-**Mod***. Then two submodule series* $0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$ *and* $0 = B_0 \leq B_1 \leq \cdots \leq B_m = B$ *are* **isomorphic** *if* $n = m$ *and there is a permutation of the indices,* σ*, such that* $A_i/A_{i-1} \cong B_{\sigma(i)}/B_{\sigma(i)-1}$ *for* $i = 1, 2, \ldots, n$*. In this situation we will say* A *and* B *have* **isomorphic submodule series** *and write* $A \sim B$.

It is clear that isomorphism of submodule series is an equivalence relation, and that if $0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$ and $0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$ are two isomorphic submodule series then any refinement of one of these series induces an isomorphic refinement of the other series.

The most important property of submodule series is the Schreier refinement theorem which says that any two submodule series in a module have isomorphic refinements. This is exactly what is needed to make $\sim$ an equivalence relation:

Proposition 3.3. $\sim$ *is an equivalence relation on* R-**Mod**.

Proof. Reflexivity and symmetry are trivial, so it remains to check transitivity...

Suppose $A \sim B$ and $B \sim C$. From the first relation we get isomorphic submodule series $0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$ and $0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$. From the second relation we get isomorphic submodule series $0 = B'_0 \leq B'_1 \leq \cdots \leq B'_m = B$ and $0 = C_0 \leq C_1 \leq \cdots \leq C_m = C$. From the Schreier refinement theorem, the two series in B have isomorphic refinements. These new isomorphic submodule series in B induce isomorphic refinements in A and C. Hence $A \sim C$. □

We will write $[A]$ for the $\sim$-equivalence class containing $A \in R$-**Mod**. Note that the zero module by itself is a $\sim$-equivalence class, that is, $[0] = \{0\}$.

If $\mathcal{S}$ is a Serre subcategory of R-**Mod** and $A \in \mathcal{S}$ then the factors of any submodule series for A are also in $\mathcal{S}$. So, in particular, if $B \in R$-**Mod** with $B \sim A$ then $B \in \mathcal{S}$. Thus $\mathcal{S}$ is a union of $\sim$-equivalence classes.

Lemma 3.4. *If* $A, B, C \in R$-**Mod***, then* $A \sim B \implies A \oplus C \sim B \oplus C$.

Proof. Let $0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$ and $0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$ be isomorphic submodule series, then it is easily checked that $0 \leq A_0 \oplus C \leq A_1 \oplus C \leq \cdots \leq A_n \oplus C = A \oplus C$ and $0 \leq B_0 \oplus C \leq B_1 \oplus C \leq \cdots \leq B_n \oplus C = B \oplus C$ are isomorphic submodule series in $A \oplus C$ and $B \oplus C$. □

This lemma has the immediate consequence that if $A \sim B$ and $C \sim D$ then $A \oplus C \sim B \oplus D$. That is, $\oplus$ induces a well defined operation on the $\sim$-equivalence classes. Since a Serre subcategory is closed under finite direct sums, $\oplus$ induces a well defined operation on the $\sim$-equivalence classes which are contained in it. We formalize this in the following definition:

Definition 3.5. *Let $\mathcal{S}$ be a Serre subcategory of R-**Mod**. We will write $M(\mathcal{S})$ for $\mathcal{S}/{\sim}$, the class of $\sim$-equivalence classes of $\mathcal{S}$, and define the operation $+$ on $M(\mathcal{S})$ by $[A] + [B] = [A \oplus B]$ for all $A, B \in \mathcal{S}$.*

$(M(\mathcal{S}), +)$ is, in fact, a commutative monoid (and, by 3.8, a refinement monoid). Rather than proving this directly we will use the following more general and useful proposition.

Proposition 3.6. *Let $\mathcal{S}$ be a Serre subcategory of R-**Mod**, N a class with a binary operation $+$, and $\Lambda\colon \mathcal{S} \to N$, a function. Then the following properties of Λ are equivalent:*

(i) *$\Lambda(B) = \Lambda(A) + \Lambda(C)$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{S}$.*
(ii) *$\Lambda(A) = \Lambda(B)$ for any $A, B \in \mathcal{S}$ with $A \sim B$, and $\Lambda(A \oplus B) = \Lambda(A) + \Lambda(B)$ for all $A, B \in \mathcal{S}$.*

If either property is true, then $\Lambda(\mathcal{S})$ is a commutative monoid with identity element $\Lambda(0)$. Also, for $A \in \mathcal{S}$, we have $\Lambda(A) = \Lambda(A_1) + \Lambda(A_2) + \cdots + \Lambda(A_n)$ where $A_1, A_2, \ldots A_n$ are the successive factors of any submodule series for A.

Proof. We show first that (i) implies (ii), and at the same time we prove the other claims of the proposition:

(1) For any $A, B \in \mathcal{S}$, the obvious exact sequence $0 \to A \to A \oplus B \to B \to 0$ implies $\Lambda(A \oplus B) = \Lambda(A) + \Lambda(B)$.
(2) Let $A \in \mathcal{S}$, then the exact sequences $0 \to A \xrightarrow{\text{id}} A \to 0 \to 0$ and $0 \to 0 \to A \xrightarrow{\text{id}} A \to 0$ imply that $\Lambda(A) + \Lambda(0) = \Lambda(A) = \Lambda(0) + \Lambda(A)$. Thus $\Lambda(0)$ is an identity of $\Lambda(\mathcal{S})$.
(3) Suppose $\sigma\colon A \to B$ is an isomorphism with $A, B \in \mathcal{S}$, then $0 \to A \xrightarrow{\sigma} B \to 0 \to 0$ is an exact sequence and so $\Lambda(B) = \Lambda(A) + \Lambda(0) = \Lambda(A)$. So we have shown that $A \cong B$ implies $\Lambda(A) = \Lambda(B)$.
(4) The commutativity and associativity of the operation $+$ on $\Lambda(\mathcal{S})$ come directly from these same properties of $\oplus$ up to isomorphism. With 2, we have proved that $\Lambda(\mathcal{S})$ is a commutative monoid with identity $\Lambda(0)$.
(5) Suppose $0 = A'_0 \leq A'_1 \leq \cdots \leq A'_n = A$ is a submodule series for $A \in \mathcal{S}$ with factors $A_i = A'_i/A'_{i-1}$. All A'_i and A_i are in $\mathcal{S}$. For each i we have the exact sequence $0 \to A'_{i-1} \to A'_i \to A_i \to 0$, so $\Lambda(A'_i) = \Lambda(A_i) + \Lambda(A'_{i-1})$. A simple induction then shows that $\Lambda(A) = \Lambda(A_1) + \Lambda(A_2) + \cdots + \Lambda(A_n)$.
(6) If $A, B \in \mathcal{S}$ have isomorphic submodule series, that is, $A \sim B$, then using 3, 4 and 5, we get $\Lambda(A) = \Lambda(B)$.

To show that (ii) implies (i), suppose $0 \to A \xrightarrow{\sigma} B \to C \to 0$ is exact for some $A, B, C \in \mathcal{S}$. Then $C \cong B/\text{im}(\sigma)$ with $\text{im}(\sigma) \cong A$, so B has the submodule series $0 \leq \text{im}(\sigma) \leq B$ with factors isomorphic to A and C. The module $A \oplus C$ has the submodule series $0 \leq A \oplus 0 \leq A \oplus C$ with these same factors, so $A \oplus C \sim B$. By (ii), $\Lambda(B) = \Lambda(A \oplus C) = \Lambda(A) + \Lambda(C)$.

□

Any function Λ which satisfies either of the conditions of this proposition will be said to **respect short exact sequences in** $\mathcal{S}$.

Since the map $A \mapsto [A]$ from $\mathcal{S}$ to $M(\mathcal{S})$ satisfies condition (ii) and is surjective, $M(\mathcal{S})$ is a commutative monoid with identity $[0]$. We note also that if A' is a submodule, factor module or subfactor module of $A \in \mathcal{S}$, then $[A'] \leq [A]$ in $M(\mathcal{S})$.

The monoid $M(\mathcal{S})$ has the following universal property:

Proposition 3.7. *Let $\mathcal{S}$ be a Serre subcategory of R-***Mod***, N a class with a binary operation $+$, and $\Lambda\colon \mathcal{S} \to N$, a map which respects short exact sequences in $\mathcal{S}$. Then Λ factors uniquely through $M(\mathcal{S})$. Specifically, there exists a unique monoid homomorphism λ from $M(\mathcal{S})$ to $\Lambda(\mathcal{S})$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{[\ ]} & M(\mathcal{S}) \\ & \searrow^{\Lambda} & \downarrow{\scriptstyle\lambda} \\ & & \Lambda(\mathcal{S}) \end{array}$$

Proof. Define the map $\lambda\colon M(\mathcal{S}) \to \Lambda(\mathcal{S})$ by $\lambda([A]) = \Lambda(A)$ for all $A \in \mathcal{S}$. This is well defined because if $[A] = [B]$, then $A \sim B$ and, by 3.6, $\Lambda(A) = \Lambda(B)$. For any $[A], [B] \in M$, we have $\lambda([A] + [B]) = \lambda([A \oplus B]) = \Lambda(A \oplus B) = \Lambda(A) + \Lambda(B) = \lambda([A]) + \lambda([B])$. Also, $\lambda([0]) = \Lambda(0)$ which is the identity for $\Lambda(\mathcal{S})$. So λ is a monoid homomorphism. □

We note that, in this proposition, if N happened to be a monoid, the homomorphism λ would not be a monoid homomorphism when viewed as a map to N unless, in addition, $\Lambda(0) = 0$. This will indeed be the case in all the applications of the proposition we will make.

Proposition 3.7 provides a second characterization of the equivalence relation $\sim$ for modules $A, B \in \mathcal{S}$, namely, $A \sim B$ if and only if the modules A and B are indistinguishable by functions on $\mathcal{S}$ which respect short exact sequences in $\mathcal{S}$.

Suppose a submodule series is given for a module A and there is another module B such that $A \sim B$, then the Schreier refinement theorem implies that there is a refinement of the existing series in A which is isomorphic to a submodule series in B. If B also happened to have a submodule series given, then a second application of the theorem would give refinements of the two given series which are isomorphic. This principle is used in showing that $M(\mathcal{S})$ is a refinement monoid:

Proposition 3.8. *For any Serre subcategory $\mathcal{S}$ of R-***Mod***, $M(\mathcal{S})$ is a refinement monoid.*

Proof. Suppose there are modules $A, B, C, D \in \mathcal{S}$ such that $[A] + [B] = [C] + [D]$ in $M(\mathcal{S})$. Then $[A \oplus B] = [C \oplus D]$, that is, $A \oplus B \sim C \oplus D$. From the above discussion, there are isomorphic submodule series for these two modules which are refinements of the series $0 \leq A \oplus 0 \leq A \oplus B$ and $0 \leq C \oplus 0 \leq C \oplus D$. That is, there are submodule series $0 \leq A_1 \leq \cdots \leq A$, $0 \leq B_1 \leq \cdots \leq B$, $0 \leq C_1 \leq \cdots \leq C$, and $0 \leq D_1 \leq \cdots \leq D$ such that the series

$$0 \leq A_1 \oplus 0 \leq \cdots \leq A \oplus 0 \leq A \oplus B_1 \leq \cdots \leq A \oplus B$$

and

$$0 \leq C_1 \oplus 0 \leq \cdots \leq C \oplus 0 \leq C \oplus D_1 \leq \cdots \leq C \oplus D$$

are isomorphic.

The permutation that matches isomorphic factors in these submodule series divides them into four types: (1) $A_i/A_{i-1} \cong C_j/C_{j-1}$; (2) $A_i/A_{i-1} \cong D_j/D_{j-1}$; (3) $B_i/B_{i-1} \cong C_j/C_{j-1}$; or (4) $B_i/B_{i-1} \cong D_jD_{j-1}$ for suitable indices i, j. If we let $W, X, Y, Z \in \mathcal{S}$ be the direct sums of the factors of type 1,2,3,4 respectively, then it easily checked that $[W]+[X] = \sum_i [A_i/A_{i-1}] = [A]$ and, similarly, $[W]+[Y] = [C]$, $[X]+[Z] = [D]$, $[Y]+[Z] = [B]$, that is, we have the refinement matrix

$$\begin{array}{cc} & \begin{array}{cc}[A] & [B]\end{array} \\ \begin{array}{c}[C]\\ [D]\end{array} & \begin{pmatrix} [W] & [Y] \\ [X] & [Z] \end{pmatrix}. \end{array}$$

□

By construction, $M(\mathcal{S})$ is a submonoid of $M(R\text{-}\mathbf{Mod})$ for each Serre subcategory $\mathcal{S}$. In fact, we have a stronger relationship:

Proposition 3.9. *For any Serre subcategory $\mathcal{S}$ of R-**Mod**, $M(\mathcal{S})$ is an order ideal of $M(R\text{-}\mathbf{Mod})$.*

Proof. Let Γ: $R\text{-}\mathbf{Mod} \to \{0, \infty\}$ be defined by

$$\Gamma(A) = \begin{cases} 0 & \text{if } A \in \mathcal{S} \\ \infty & \text{if } A \notin \mathcal{S} \end{cases}$$

It is easily checked that Γ respects short exact sequences in R-**Mod**, so Proposition 3.7 provides a monoid homomorphism γ: $M(R\text{-}\mathbf{Mod}) \to \{0, \infty\}$ such that $\Gamma(A) = \gamma([A])$ for all $A \in R\text{-}\mathbf{Mod}$. Hence $M(\mathcal{S}) = \gamma^{-1}(0)$ is an order ideal of $M(R\text{-}\mathbf{Mod})$. □

In particular, $M(R\text{-}\mathbf{Noeth})$ is an order ideal of $M(R\text{-}\mathbf{Mod})$.

4. The Krull Length of a Module

The remaining ingredient that we need for the proof of the main theorems of this paper is the Krull length function. This function is an extension of both the composition series length and the Krull dimension.

We will write **Ord** for the class of ordinals, $\mathbf{Ord}^*$ for $\mathbf{Ord} \cup \{-1\}$, and $\mathrm{Kdim}(A) \in \mathbf{Ord}^*$ for the Krull dimension of a module A in the sense of Gordon and Robson [6], when this dimension exists. The Krull dimension of the zero module is defined to be -1. The basic properties of the Krull dimension are

Proposition 4.1.

(1) $\mathrm{Kdim}(A)$ *exists for any Noetherian module A.*
(2) *If $0 \to A \to B \to C \to 0$ is a short exact sequence of modules with Krull dimension, then*

$$\mathrm{Kdim}(B) = \max\{\mathrm{Kdim}(A), \mathrm{Kdim}(C)\}.$$

Proof. See [5, Lemma 13.3]. □

Definition 4.2. *Let $\alpha \in$ **Ord**. A module A is α-**critical** if $\mathrm{Kdim}(A) = \alpha$ and $\mathrm{Kdim}(A/B) < \alpha$ for all nonzero submodules B of A. A module is **critical** if it is α-critical for some ordinal α.*

The following are simple consequences of the definition and 4.1:

Proposition 4.3.

(1) *Any nonzero submodule of an α-critical module is α-critical.*
(2) *Let A be an α-critical module with submodule series*

$$0 = A_0 < A_1 \leq \ldots \leq A_n = A.$$

Then A_1 is an α-critical module and $\operatorname{Kdim}(A_i/A_{i-1}) < \alpha$ *for* $i = 2, 3, \ldots, n$.

From 2 we note that any submodule series for an α-critical module has exactly one α-critical factor.

Proposition 4.4. *If A is a nonzero Noetherian module, then A has a submodule series*

$$0 = A_0 \leq A_1 \leq \ldots \leq A_n = A,$$

in which all factors A_i/A_{i-1} are critical modules.

Proof. See [5, Theorem 13.9]. □

Let A be a nonzero Noetherian module with $\alpha = \operatorname{Kdim}(A)$ and submodule series as provided by the above proposition. Then since

$$\operatorname{Kdim}(A) = \max\{\operatorname{Kdim}(A_i/A_{i-1}) \mid i = 1, 2, \ldots, n\},$$

all factors must have Krull dimension less than or equal to α and there must be at least one α-critical factor.

Proposition 4.5. *Let A be a nonzero Noetherian module, with $\alpha = \operatorname{Kdim}(A)$. Then the number of α-critical factors in a submodule series*

$$0 = A_0 \leq A_1 \leq \ldots \leq A_n = A,$$

in which all factors A_i/A_{i-1} are critical modules, is independent of the choice of submodule series.

Proof. Suppose there is a second submodule series,

$$0 = A'_0 \leq A'_1 \leq \ldots \leq A'_m = A,$$

whose factors are all critical modules. By the Schreier refinement theorem, these two series have isomorphic refinements. Using 4.3.2, there are exactly as many α-critical factors in each of the refinements as in the original series. □

Definition 4.6. *For a nonzero Noetherian module A we define the* **Krull length** *of A by*

$$\operatorname{Klen}(A) = (\alpha, n) \in \mathbf{Ord} \times \mathbb{N}$$

where $\alpha = \operatorname{Kdim}(A)$ and n is the number of α-critical factors in a submodule series for A whose factors are critical. From the above proposition, Klen *is well defined.*

The Krull length function can also be considered an extension of the functions discussed by G. Krause in [7]. In his notation, $\operatorname{Klen}(A) = (\alpha, \lambda_\alpha(A))$ if $\operatorname{Kdim}(A) = \alpha$.

Proposition 4.7. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of nonzero Noetherian modules with* $\operatorname{Klen}(A) = (\alpha, m)$ *and* $\operatorname{Klen}(C) = (\gamma, n)$. *Then*

$$\operatorname{Klen}(B) = \begin{cases} (\alpha, m) & \text{if } \gamma < \alpha \\ (\gamma, n) & \text{if } \alpha < \gamma \\ (\alpha, m+n) & \text{if } \alpha = \gamma \end{cases}$$

Proof. Without loss of generality, we can assume that A is a submodule of B and $C = B/A$.

For each of A and C there is a submodule series with critical factors. These can be concatenated to form a submodules series for B,

$$0 = A_0 \leq A_1 \leq \ldots \leq A_n = A = C_0 \leq C_1 \leq \ldots \leq C_m = B$$

with critical factors.

We have $\beta = \mathrm{Kdim}(B) = \max\{\alpha, \gamma\}$, and the number of β-critical factors in this series is the sum of the number of β-critical factors in each of the series for A and C. The claim then follows easily.

□

To make the function Klen into a function on R-**Noeth** which respects short exact sequences, we are led to define the Krull length of the zero module to be the symbol 0 and construct a monoid as the image of Klen:

Definition 4.8. *We define the monoid* **Krull** *as follows:*

As a class, $\mathbf{Krull} = (\mathbf{Ord} \times \mathbb{N}) \cup \{0\}$. *The operation* $+$ *is given by*

(i) $0 + 0 = 0$

(ii) $0 + (\alpha, n) = (\alpha, n) + 0 = (\alpha, n)$ *for all* $(\alpha, n) \in \mathbf{Ord} \times \mathbb{N}$

(iii)
$$(\alpha, m) + (\gamma, n) = \begin{cases} (\alpha, m) & \text{if } \gamma < \alpha \\ (\gamma, n) & \text{if } \alpha < \gamma \\ (\alpha, m+n) & \text{if } \alpha = \gamma \end{cases}$$

for all $(\alpha, n), (\gamma, m) \in \mathbf{Ord} \times \mathbb{N}$.

It is easily checked that **Krull** *is a commutative monoid, whose preorder, when restricted to* $\mathbf{Ord} \times \mathbb{N}$, *coincides with the lexicographic order. In particular,* **Krull** *is Artinian. We also have that* $2x \leq x$ *in* **Krull** *if and only if* $x = 0$.

By construction, the map Klen: R-**Noeth** $\to$ **Krull** respects short exact sequences in R-**Noeth**, so, by 3.7, there is an induced monoid homomorphism from $M(R\text{-}\mathbf{Noeth})$ to **Krull**. The induced map we will also call Klen, that is, we define $\mathrm{Klen}([A]) = \mathrm{Klen}(A)$ for all $A \in R\text{-}\mathbf{Noeth}$. An element $[A] \in M(R\text{-}\mathbf{Noeth})$ satisfies $\mathrm{Klen}([A]) = 0$ if and only if $[A] = 0$. The property of this map that we need for applying 2.9 to $M(R\text{-}\mathbf{Noeth})$ is

$$\begin{aligned} \mathrm{Klen}(2[A]) \leq \mathrm{Klen}([A]) &\implies 2\mathrm{Klen}([A]) \leq \mathrm{Klen}([A]) \\ &\implies \mathrm{Klen}([A]) = 0 \\ &\implies [A] = 0. \end{aligned}$$

5. Main Results

We now have all the ingredients in place to apply Theorem 2.9 to the monoid $M(R\text{-}\mathbf{Noeth})$:

Theorem 5.1. *The monoid* $M(R\text{-}\mathbf{Noeth})$ *is strongly separative.*

Proof. We have the monoid homomorphism Klen: $M(R\text{-}\mathbf{Noeth}) \to \mathbf{Krull}$ such that for any $[A] \in M(R\text{-}\mathbf{Noeth})$, $\mathrm{Klen}(2[A]) \leq \mathrm{Klen}([A])$ implies $[A] = 0$. Since $M(R\text{-}\mathbf{Noeth})$ has refinement and **Krull** is Artinian, Theorem 2.9 implies that $M(R\text{-}\mathbf{Noeth})$ is strongly separative. □

This theorem is proved using Theorem 2.9 rather than Corollary 2.10 because the monoid $M(R\text{-}\mathbf{Noeth})$ is not, in general, Artinian. See [3] for details.

Theorem 5.1 has a lot of consequences for Noetherian modules which can be obtained by reinterpreting a relationship among modules as an equation in the monoid $M(R\text{-}\mathbf{Noeth})$, and then applying strong separativity.

For example, the existence of any of the following types of exact sequences in R-**Noeth** implies that $A \sim B$:

$$0 \to A \to A \oplus B \to A \to 0,$$

$$0 \to A \to A \oplus A \to B \to 0,$$

$$0 \to A \oplus A \to A \oplus A \oplus B \to A \to 0.$$

We will prove this claim for the last example: From the given short exact sequence we get

$$[A \oplus A \oplus B] = [A \oplus A] + [A],$$

and so $2[A] + [B] = 3[A]$ in $M(R\text{-}\mathbf{Noeth})$. Since $M(R\text{-}\mathbf{Noeth})$ is strongly separative, we can apply 2.3.5 with $a = [A]$, $b = [B]$ and $n = 2$ to get $[A] = [B]$, that is, $A \sim B$.

For longer exact sequences we have the rule that if

$$0 \to A \to B \xrightarrow{\beta} C \xrightarrow{\gamma} D \to 0$$

is an exact sequence in R-**Mod**, then $[A] + [C] = [B] + [D]$. This is proved by making the two short exact sequences

$$0 \to A \to B \xrightarrow{\beta} \operatorname{im}(\beta) \to 0,$$

and

$$0 \to \ker(\gamma) \to C \xrightarrow{\gamma} D \to 0.$$

Since $\operatorname{im}(\beta) = \ker(\gamma)$, we get $[A]+[C] = [A]+[\ker(\gamma)]+[D] = [A]+[\operatorname{im}(\beta)]+[D] = [B] + [D]$.

With this rule and 5.1, the existence of either of the following types of exact sequences in R-**Noeth** (among many others) implies that $A \sim B$:

$$0 \to A \to A \to A \to B \to 0,$$

$$0 \to A \to B \to A \to A \to 0.$$

We can also apply Theorem 5.1 in a similar way to direct sums of Noetherian modules. For example, if $A, B \in R$-**Noeth**, then

$$A \oplus A \sim A \oplus B \implies A \sim B.$$

We make a simple illustrative application of this result to Weyl algebras:

Corollary 5.2. *Let $R = A_1(k)$ be the first Weyl algebra over a field k of characteristic 0. Then for any nonzero left ideal $I \leq R$ we have $I \sim R$.*

Proof. The ring R is left Noetherian and from [11, Theorem 2] we have $I \oplus R \cong R \oplus R$. ☐

This corollary in some sense repairs the fact that $A_1(k)$ fails to be a principal ideal domain.

Since $M(R\text{-}\mathbf{Noeth})$ is a strongly separative order ideal in $M(R\text{-}\mathbf{Mod})$, we can use 2.7 to get stronger cancellation properties which involve modules which are not Noetherian. For example, if

$$0 \to C \to A \to B \to C \to 0$$

is an exact sequence in R-**Mod** with $C \in R$-**Noeth**, then $A \sim B$. Here we use the fact that C is isomorphic to a submodule of A and so $[C] \le [A]$.

For comparison with Theorem 5.5 we single out one particular result of this type derived from 2.7.3:

Corollary 5.3. *Let $A, B \in R$-**Mod**, $C \in R$-**Noeth** and $n \in \mathbb{N}$. If C is a submodule, factor module or subfactor module of $\bigoplus^n (A \oplus B)$, and $A \oplus C \sim B \oplus C$, then $A \sim B$.*

The final aim of this paper is to show that, in this corollary, we can drop the hypothesis on C if we have $A \oplus C \cong B \oplus C$ instead of $A \oplus C \sim B \oplus C$. To do this we need a way of cutting down the size of C in the relation $A \oplus C \cong B \oplus C$ so that C is comparable to $A \oplus B$:

For any R-module X we define a map T_X: R-**Mod** $\to$ R-**Mod** by

$$T_X(C) = \sum \{\mathrm{im}(\gamma) \mid \gamma \in \mathrm{Hom}_R(X, C)\},$$

that is, $T_X(C)$ is the sum of all submodules of C which are isomorphic to factor modules of X. We note that if X_1 is a direct summand of X then $T_X(X_1) = X_1$.

Lemma 5.4. *For all $C_1, C_2, X \in R$-**Mod**, $T_X(C_1 \oplus C_2) = T_X(C_1) \oplus T_X(C_2)$.*

Proof. See [1, Proposition 8.18]. □

Theorem 5.5. *If $A, B \in R$-**Mod** and $C \in R$-**Noeth** such that $A \oplus C \cong B \oplus C$, then $A \sim B$.*

Proof. We apply the map $T_{A\oplus B}$ to the equation $A \oplus C \cong B \oplus C \ldots$

We have $T_{A\oplus B}(A) = A$ and $T_{A\oplus B}(B) = B$, so using 5.4, we get $A \oplus C' \cong B \oplus C'$ where $C' = T_{A\oplus B}(C)$.

The module C is Noetherian, so C' is a finite sum of images of $A \oplus B$, that is, there is an $n \in \mathbb{N}$ such that C' is a factor module of $\bigoplus^n (A \oplus B)$. Since C' is Noetherian, 5.3 implies that $A \sim B$. □

We should remark that this theorem is not true with the weaker hypothesis that $A \oplus C \sim B \oplus C$. For example, let $R = \mathbb{Z}$. Then from the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

we get $[\mathbb{Z}] = [\mathbb{Z}/2\mathbb{Z}] + [\mathbb{Z}] = [(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}]$. Hence $0 \oplus \mathbb{Z} \sim (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$ but $0 \not\sim \mathbb{Z}/2\mathbb{Z}$.

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