Pascal's Labeling and Path Counting

Ko-Wei Lih
Institute of Mathematics
Academia Sinica
Nankang, Taipei 115, Taiwan
E-mail: makwlih@sinica.edu.tw

Daphne Der-Fen Liu
Department of Mathematics
California State University, Los Angeles
Los Angeles, CA 90032, U. S. A.
E-mail: dliu@calstatela.edu

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The purpose of this expository paper is to achieve the following objectives in a self-contained manner.

1. Construct Pascal's labeling of lattice points of Euclidean plane.
2. Show that labels are equal to binomial coefficients.
3. Show that binomial coefficients or their absolute values count certain lattice paths.
4. Demonstrate that Pascal's labeling makes it easier to discover and prove identities involving binomial coefficients.

1 Pascal's labeling

We first draw a Cartesian coordinate mesh on the two dimensional Euclidean plane so that each intersection point has integer coordinates. We call these intersection points *lattice points*. We want to label lattice points in an orderly fashion so that each label is determined by two of its neighboring labels. We proceed with the following rule. If the label given to a lattice point with coordinates (n, k) is denoted by $[n, k]$, then we stipulate that

$$[n, k] = [n - 1, k] + [n - 1, k - 1]. \tag{1}$$

This identity says that any label is equal to the sum of the labels immediately to its left and to its lower left.

In addition to the above rule, we need some initial labels to get the process started. Since any one term of (1) is completely determined by the other two, it is easy to see that, once labels on the horizontal and the upper vertical axes are fixed, labels of all lattice points can be computed by means of (1). We could choose any real numbers as initial labels. But we are labeling points with integer coordinates, it seems natural that we use

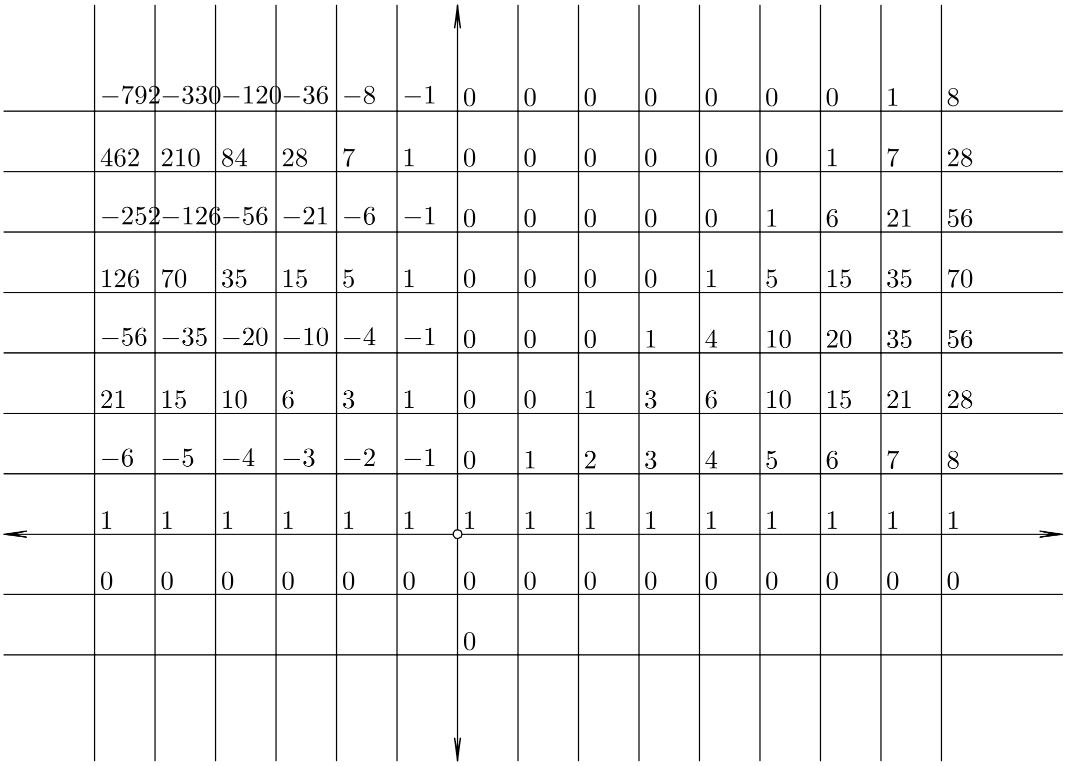

Figure 1: Pascal's labeling

integers as initial labels. The simplest choice would be making everything 0. Then we get a whole plane of 0's which is a bit dull. To make things more interesting, we label all lattice points on the horizontal axis by 1 and all lattice points on the upper vertical axis by 0. The outcome of the labeling process is illustrated in Figure 1. We call it *Pascal's* labeling of the plane.

2 ENE-paths and the binomial coefficient

2.1 Counting ENE-paths

What are these labels? Do they have special meanings? Since they are all nonnegative numbers in the first quadrant, do they count some sort of objects? In order to answer these questions, we first pay our attention only to the first quadrant.

Let us consider paths starting at the origin that move according to the following rules. A path can stay at the origin without going anywhere. It is a "trivial" path that starts at the origin and ends at the origin. For a non-trivial path, there are two kinds of admissible steps going out of any lattice point: either it moves to the right neighboring lattice point (called an E-step) or it moves to the north-east neighboring lattice point (called an NE-step). Any path that is determined by these rules is called an *ENE-path*. An ENE-path from the origin to the point $(6, 3)$ is illustrated in Figure 2.

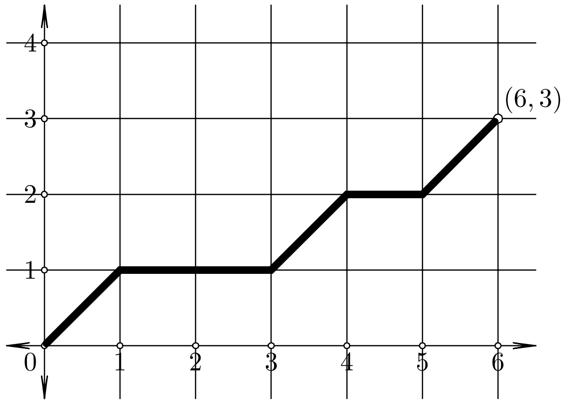

Figure 2: An ENE-path

An ENE-path terminating at a point (n, k) comes either from an ENE-path terminating at the point $(n-1, k)$ or from an ENE-path terminating at the point $(n-1, k-1)$. So the number of ENE-paths terminating at the point (n, k) satisfies precisely the same identity (1).

How many ENE-paths terminate at the origin? One, the trivial one. How many ENE-paths terminate at a point on the upper vertical axis? None, the first step of an ENE-path can never go vertical. How many ENE-paths terminate at a point on the positive horizontal axis? One, every step of that path must move horizontally. The numbers of ENE-paths terminating at the nonnegative axes are precisely the initial labels given by Pascal's labeling.

With the same initial numbers and the same defining formula, the obvious conclusion is as follows.

The label given to a lattice point (n, k) of the first quadrant in Pascal's labeling counts the number of ENE-paths terminating at that point.

In this way, we have given a "combinatorial" interpretation to those labels, well, at least in the first quadrant. Of course, this interpretation can be extended in a trivial way to the fourth quadrant filled with 0's because none of the ENE-paths are going there.

2.2 Binomial coefficients as labels

Now we have understood what those labels "really" mean. How are we going to compute them once a point (n, k) is given. Of course, we can start with initial labels and work our way to (n, k) by means of (1) in every step. That surely guarantees us to get the correct label. But it is too cumbersome when a point is far away from the origin. Is there a compact formula to compute those labels in a straightforward manner?

Let us take a closer look into how an ENE-path is determined. Suppose that a point (n, k) is given, where both n and k are positive integers. An ENE-path from the origin is going to move n steps to reach that point. We may use a sequence $(a_1, a_2, \ldots, a_n)$ of length n to record its movement. Each a_i is either 0 or 1 indicating that the i-th step is an E-step or an NE-step, respectively. We call such a sequence the *signature* sequence of that ENE-path. For example, in Figure 2 the path has signature sequence $(1, 0, 0, 1, 0, 1)$. Since each ENE-path climbs up exactly k units to reach (n, k), there are exactly k 1's among the a_i's. Conversely, each sequence of k 1's and $n-k$ 0's completely determines an ENE-path terminating at (n, k). So, if we want to count how many ENE-paths terminating at the point (n, k), we might as well count such sequences instead.

How do we count signature sequences? We start with a sequence of n empty positions. We pick a position to put the first 1 there. There are obviously n choices for this first position. When we continue to pick another position to place the second 1, there are only $n-1$ empty positions left to choose. Totally there are $n(n-1)$ ways to place the first two 1's. If we keep doing this way, then each time we reduce the possible choice for empty positions by one. The final count will be $n(n-1)(n-2)\cdots(n-k+1)$ possible choices for placing k 1's. However, all these possible choices will not produce distinct sequences. The reason is because, once the k positions for placing 1's are fixed, the sequence produced is the same no matter in what order we fill in those 1's. Since there are $k! = k(k-1)(k-2)\cdots 2\cdot 1$ ways of fixing an ordering of these k 1's, there are only $n(n-1)(n-2)\cdots(n-k+1)/k!$ distinct legitimate signature sequences. We use a special notation $\binom{n}{k}$ to denote this number and conclude that the formula

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

computes the number of ENE-paths reaching the point (n, k).

So far this notation $\binom{n}{k}$ only makes sense when both n and k are positive integers. We know that there is a trivial ENE-path from the origin to itself. So we make the convention $\binom{0}{0} = 1$. Moreover, we observe that the numerator of $\binom{n}{k}$ even makes sense when n is any real number. We just go ahead to extend the definition of $\binom{n}{k}$ by the following formula, in which r denotes an arbitrary real number.

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\cdots(r-k+1)}{k!}, & \text{integer } k > 0; \\ 1, & k = 0; \\ 0, & \text{integer } k < 0. \end{cases} \tag{2}$$

For instance, when $r = \frac{5}{2}$,

$$\binom{\frac{5}{2}}{2} = \frac{\frac{5}{2}(\frac{5}{2}-1)}{2} = \frac{15}{8}.$$

These numbers $\binom{r}{k}$'s are called *binomial coefficients* because they appear as coefficients in the expansion of $(x+y)^r$ when r is a nonnegative integer or $|x/y| < 1$. If r and k are positive integers, then formula (1) expressed in terms of binomial coefficients looks like the following.

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}. \tag{3}$$

We call this formula the *basic recursion*. It is called a recursion because it computes the value on the left-hand side with the help of earlier values on the right hand side, which are supposed to have been computed in previous stages. Does the basic recursion hold in the other three quadrants of the plane? Let us verify it by manipulating the definition.

$$\begin{aligned}
\binom{r-1}{k} + \binom{r-1}{k-1} &= \frac{(r-1)\cdots(r-k)}{k!} + \frac{(r-1)\cdots(r-k+1)}{(k-1)!} \\
&= \frac{(r-1)\cdots(r-k)}{k!} + \frac{(r-1)\cdots(r-k+1)k}{k!} \\
&= \frac{(r-1)\cdots(r-k+1)(r-k+k)}{k!} \\
&= \frac{r(r-1)\cdots(r-k+1)}{k!} = \binom{r}{k}.
\end{aligned}$$

The above verification works for $k \geqslant 2$. The cases for $k \leqslant 1$ can be checked without any difficulty.

Now the binomial coefficients satisfy recursion (3) for all permissible arguments and they coincides with our labels on the axes, hence they are identical with our labels on all lattice points. As a matter of fact, any point (r, k) (not necessarily a lattice point) on any horizontal line in Figure 1 can have a label $\binom{r}{k}$.

3 Symmetries

3.1 Symmetries in the first quadrant

We have succeeded in interpreting Pascal's labeling as marking points with binomial coefficients. However, a casual inspection of Figure 1 will reveal some regularities and symmetries. Such phenomena should lead to identities involving binomial coefficients.

We first observe that all labels along the main 45° diagonal line in the first quadrant are 1's. This certainly holds in general since a non-trivial ENE-path must take every step in the northeast direction to reach a point on the main diagonal. This offers a very simple proof for the identity

$$\binom{n}{n} = 1, \quad \text{integer } n \geqslant 0.$$

The main reason that we called our labeling of the plane Pascal's labeling is because the part below the main diagonal is merely a different depiction of the following array of numbers, commonly known as Pascal's triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \dots & & \dots & & \dots & & \dots & & \dots &
\end{array}
$$

Pascal's triangle obscures the possibility of extending binomial coefficients to negative arguments and makes it hard to visualize symmetries, except the obvious left-right symmetry with respect to the central vertical line. We should shed the historical habit of exhibiting binomial coefficients in Pascal's triangle and start using Pascal's labeling more extensively.

When the left-right symmetry of Pascal's triangle is transplanted into our Pascal's labeling of the plane, it shows that, in the first quadrant, the label found at k units below the main diagonal coincides with the label found at k units above the horizontal axis. We should be able to establish the following identity:

$$\binom{n}{k} = \binom{n}{n-k}, \quad \text{integer } n \geqslant 0, \text{ integer } k. \tag{4}$$

To prove the above is simple since we can switch the 0's and 1's of the signature sequence of an ENE-path to the point (n, k) to get the signature sequence of an ENE-path to the point $(n, n-k)$, and vise versa. The identity holds trivially if k is negative.

The above identity also expresses an orderly correspondence between lines. When we fix the nonnegative k and let n vary, all labels $\binom{n}{k}$ are arranged along a horizontal line through the point $(0, k)$. Since every label $\binom{n}{n-k}$ has a fixed difference between the upper and the lower labels, all these labels are arranged along a 45° diagonal line through the point $(0, -k)$. Therefore identity (4) matches up the corresponding labels on those two lines.

3.2 WN-paths and symmetries in the second quadrant

Since a large portion of binomial coefficients in the second quadrant are negative, we may have the impression that they do not count objects. However, if we consider the absolute values of those binomial coefficients, we may obtain an appropriate combinatorial interpretation.

Let us call the second quadrant with the nonnegative vertical axis removed the *reduced* second quadrant. We see that in this reduced second quadrant the absolute values of binomial coefficients form a Pascal's triangle with the tip of the triangle placed at the point $(-1, 0)$.

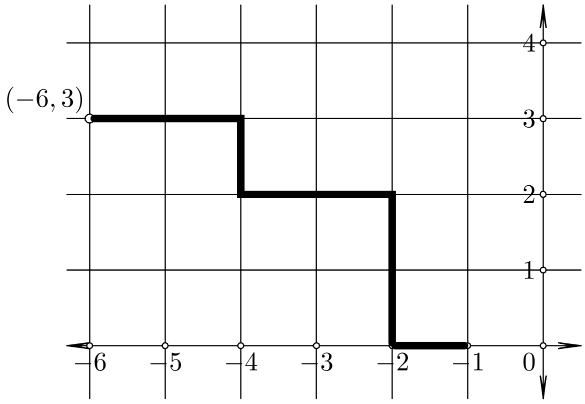

Figure 3: A WN-path

How are we going to give a path-counting interpretation of these absolute values?

The Pascal's triangle in the reduced second quadrant can be regarded as a "fan-out" of the Pascal's triangle in the first quadrant situated between the main diagonal line and the horizontal axis. If we are prepared to flip the path-counting apparatus from the first quadrant to the second quadrant, we have to change the orientation of a path properly. It is not hard to recognize that all NE-steps should correspond to upward going *N-steps* and E-steps should correspond to leftward going *W-steps*. More formally, we say that a path is a *WN-path* if it starts at the point $(-1, 0)$ and each of its movement falls into one of the following two types: either it moves to the left neighboring lattice point (called a W-step) or it moves to the north neighboring lattice point (called an N-step). A WN-path is illustrated in Figure 3.

When $n < 0$ and $k \geqslant 0$ are integers, we use $\left\{ n \atop k \right\}$ to denote the number of WN-paths from the point $(-1, 0)$ to the point (n, k). Clearly, the following recursion is satisfied.

$$\left\{ n-1 \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + \left\{ n \atop k \right\}. \tag{5}$$

Now we consider the term $(-1)^k \left\{ n \atop k \right\}$. We see that

$$\begin{aligned} & (-1)^k \left\{ n-1 \atop k \right\} + (-1)^{k-1} \left\{ n-1 \atop k-1 \right\} \\ = \ & (-1)^k (\left\{ n-1 \atop k \right\} - \left\{ n-1 \atop k-1 \right\}) \\ = \ & (-1)^k \left\{ n \atop k \right\}, \qquad \text{by (5).} \end{aligned}$$

This shows that $(-1)^k \left\{ n \atop k \right\}$ satisfies the basic recursion (3), too. As to the boundary values, we see that $(-1)^k \left\{ -1 \atop k \right\} = (-1)^k = \binom{-1}{k}$ and $(-1)^0 \left\{ n \atop 0 \right\} = 1 = \binom{n}{0}$. For integers $n < 0$ and $k \geqslant 0$, we see that

$$\binom{n}{k} = (-1)^k \left\{ n \atop k \right\}, \quad \text{or equivalently,} \quad \left\{ n \atop k \right\} = (-1)^k \binom{n}{k}. \tag{6}$$

The above discussion leads us to the following conclusion.

The absolute value of the label given to a lattice point (n, k) of the reduced second quadrant in Pascal's labeling counts the number of WN-paths terminating at that point. The parity of k gives the parity of that label.

Again, this conclusion can be extended trivially to the third quadrant filled with 0's because no WN-paths are going there. Armed with this interpretation, we are able to offer path-counting proofs. For example, now we can establish an identity analogous to (4). Note that the points (n, k) and $(-k-1, -n-1)$ are symmetric about the line determined by the equation $x + y + 1 = 0$ when $n < 0$ and $k \geqslant 0$. By interchanging W-steps with N-steps, a WN-path to the point (n, k) is converted into a WN-path to the point $(-k-1, -n-1)$, and vice versa. Hence

$$\left\{ {n \atop k} \right\} = \left\{ {-k-1 \atop -n-1} \right\}, \quad \text{integer } n < 0, \text{ integer } k \geqslant 0. \tag{7}$$

When we fix a nonnegative k and let n vary, identity (7) reveals a symmetry between a horizontal line and a vertical line both in the second quadrant.

Since $\left\{ {n \atop k} \right\} = (-1)^k \binom{n}{k}$ and $\left\{ {-k-1 \atop -n-1} \right\} = (-1)^{-n-1} \binom{-k-1}{-n-1}$, we have the following equivalent form for (7) in terms of binomial coefficients.

$$\binom{n}{k} = (-1)^{k-n-1} \binom{-k-1}{-n-1}, \quad \text{integer } n < 0, \text{ integer } k \geqslant 0. \tag{8}$$

3.3 Symmetries between the first two quadrants

We observe that there are equal numbers of 0's to the left and to the right of the point $P(\frac{k-1}{2}, k)$ on the horizontal line L_1 through the point $(0, k)$. The two lattice points on L_1 at equal distance to the point P have labels of either identical or opposite signs, depending on the parity of k. We may say that this is a *signed* symmetry. Suppose that the point to the right of P has coordinates (n, k). Then the symmetric point will have coordinates (x, k) such that $n - \frac{k-1}{2} = \frac{k-1}{2} - x$, i.e., $x = k - n - 1$. Our observation leads us to the following identity.

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad \text{integers } n \geqslant 0, \text{ integer } k. \tag{9}$$

Let us prove it. This identity is trivially true when $k < 0$. So assume that $k \geqslant 0$. The term on the left-hand side counts the number of ENE-paths to the point (n, k). If we change every NE-step to a step going vertically, then each such path is converted into a path from the origin to the point $(n - k, k)$ using only E-steps and N-steps. We flip this path to the second quadrant and translate it horizontally to the left by one unit. We finally obtain a WN-path to the point $(k - n - 1, k)$. The whole process can be reversed to get the original ENE-path back. Consequently, we have the equality

$$\binom{n}{k} = \left\{ {k-n-1 \atop k} \right\}.$$

This together with (6) imply (9).

By combining identities (4) and (9), we get the following identity.

$$\binom{n}{k} = (-1)^{n-k}\binom{-k-1}{n-k}, \quad \text{integer } n \geqslant 0, \text{ integer } k. \tag{10}$$

When we fix a nonnegative k and let n vary, identity (10) reveals a signed symmetry between a horizontal line in the first quadrant and a vertical line in the second quadrant.

Actually, we can derive more from (9). If we replace r in the defining formula (2) by a variable x, we get the following polynomial of degree k.

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}, \quad \text{integer } k > 0.$$

Similarly, $(-1)^k\binom{k-x-1}{k}$ is a polynomial of degree k. According to identity (9), these two polynomials have identical values when x runs through all nonnegative integers. It follows that they are identical polynomials and identity (9) can be generalized to the following form for any real number r.

$$\binom{r}{k} = (-1)^k\binom{k-r-1}{k}, \quad \text{integer } k. \tag{11}$$

This useful proof technique of extending identities from integers to reals is called the *polynomial argument*.

4 Identities by recursion

4.1 Alternating sums

Each ENE-path of length n will reach a unique lattice point on the vertical line segment between the point $(n, 0)$ and (n, n). Therefore the sum $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ represents the total number of ENE-paths of length n. Since there are two choices for such a path to continue at any intermediate point, the total number should be 2^n. Consequently, we have proved the following identity.

$$\sum_k \binom{n}{k} = 2^n, \quad \text{integer } n \geqslant 0. \tag{12}$$

Although the index k ranges over all integers in the above identity, there are actually finitely many nonzero summands.

If we consider WN-paths instead, then each such path of length n will reach a unique lattice point on the 45° line segment connecting the two points $(-n-1, 0)$ and $(-1, n)$. Again, since there are two choices for a WN-path to continue at any intermediate point, the total number is 2^n. Consequently, we have the following identity.

$$\sum_k \left\{ {-n-1+k \atop k} \right\} = 2^n, \quad \text{integer } n \geqslant 0.$$

In terms of binomial coefficients, it becomes

$$\sum_k (-1)^k \binom{-n-1+k}{k} = 2^n, \quad \text{integer } n \geqslant 0.$$

Of course, this can also be obtained from (12) by an application of (9).

Now instead of summing up all terms $\left\{ {-n-1+k \atop k} \right\}$ over k, we consider the sum of their corresponding binomial coefficients $\binom{-n-1+k}{k}$. By (9) again, we are actually looking for the sum

$$\sum_k (-1)^k \binom{n}{k}.$$

For $n > 0$, we may use our basic recursion to do cancellations as follows.

$$\begin{aligned}
& \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} \\
= \; & \binom{n-1}{-1} + \binom{n-1}{0} - \binom{n-1}{0} - \binom{n-1}{1} + \binom{n-1}{1} + \binom{n-1}{2} - \\
& \cdots + (-1)^n \binom{n-1}{n} \\
= \; & \binom{n-1}{-1} + (-1)^n \binom{n-1}{n} \\
= \; & 0 + 0 = 0.
\end{aligned}$$

However, the sum should be 1 if $n = 0$. If we adopt the convention that $0^0 = 1$, then the following identity is established.

$$\sum_k (-1)^k \binom{n}{k} = 0^n, \quad \text{integer } n \geqslant 0.$$

Exactly the same method used in the above proof can establish the following slightly more general result.

$$\sum_{k \leq n} (-1)^k \binom{r}{k} = (-1)^n \binom{r-1}{n}, \quad \text{integer } n.$$

Whenever the basic recursion is employed in a derivation, the proof can usually be written as a mathematical induction. Nevertheless, a proof by induction is less informative than a direct derivation because the former often leaves us wondering how the solution was obtained in the first place.

4.2 Sums along horizontal or diagonal lines

Suppose that we start at a point $(n+1, m+1)$. One application of the basic recursion makes its label $\binom{n+1}{m+1}$ equal to the sum of one label to the left and one label to the lower left. If we apply the basic recursion again to the left label, then the same phenomenon occurs. We can keep iterating the basic recursion to compute the sum of labels along a horizontal line as follows.

$$\begin{aligned}\binom{n+1}{m+1} &= \binom{n}{m+1} + \binom{n}{m} \\ &= \binom{n-1}{m+1} + \binom{n-1}{m} + \binom{n}{m} \\ &= \binom{n-2}{m+1} + \binom{n-2}{m} + \binom{n-1}{m} + \binom{n}{m} \\ &= \quad \cdots \\ &= \binom{j}{m+1} + \sum_{j \leqslant k \leqslant n} \binom{k}{m}.\end{aligned}$$

When $n \geqslant 0$ and $j = 0$, we have the following formula for summation on the upper index.

$$\sum_{0 \leqslant k \leqslant n} \binom{k}{m} = \binom{n+1}{m+1}, \quad \text{integers } m, n \geqslant 0. \tag{13}$$

On the other hand, if we keep iterating the basic recursion on the lower left labels, we get a sum along the 45° diagonal line. This summing process will be winding down to the lower half plane of 0's. In the following formula, there is no need to put an upper bound on the summing index. The summation is actually computed over finitely many nonzero terms.

$$\sum_{0 \leqslant i} \binom{n-i}{m+1-i} = \binom{n+1}{m+1}, \quad \text{integers } m, n.$$

Or equivalently,

$$\sum_{0 \leqslant i} \binom{n-i}{m-i} = \binom{n+1}{m}, \quad \text{integers } m, n.$$

This identity can be written in yet another equivalent form. We replace $m - i$ by a new variable k and rename $n - m$ as r. Then we have the following identity which holds for all reals r by the polynomial argument.

$$\sum_{k \leqslant m} \binom{r+k}{k} = \binom{r+m+1}{m}, \quad \text{integer } m. \tag{14}$$

4.3 Fibonacci numbers revealed

Now suppose that we are summing up labels from the upper left to the lower right. We choose to restrict the summation to the first and the fourth quadrants so that there are only finitely many nonzero terms on such an "anti-diagonal" line. Let

$$F_n = \sum_{0 \leqslant k} \binom{k}{n-k}, \quad \text{integer } n \geqslant 0.$$

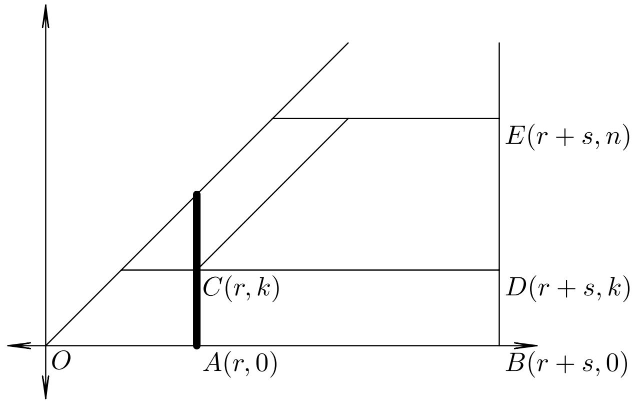

Figure 4: First Vandermonde's convolution

It is easy to see that $F_0 = F_1 = 1$. The terms F_n's satisfy the recursion

$$F_n = F_{n-1} + F_{n-2} \quad \text{for integer } n \geqslant 2 \tag{15}$$

since

$$\begin{aligned} F_n &= \binom{0}{n} + \sum_{1 \leqslant k} \binom{k}{n-k} \\ &= \sum_{1 \leqslant k} \binom{k-1}{n-k} + \sum_{1 \leqslant k} \binom{k-1}{n-k-1} \\ &= \sum_{0 \leqslant k} \binom{k}{(n-1)-k} + \sum_{0 \leqslant k} \binom{k}{(n-2)-k} \\ &= F_{n-1} + F_{n-2}. \end{aligned}$$

With the two initial values and the recursion (15), the terms F_n's are precisely the well-known Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$.

5 Vandermonde's convolutions

5.1 Basic convolutions

In this subsection, we use the method of counting ENE-paths to derive two important identities commonly known as *Vandermonde's convolutions*.

Given positive integers r, s, and n, the number of ENE-paths from the origin to the point $E(r+s, n)$ in Figure 4 is equal to $\binom{r+s}{n}$. Each such ENE-path intersects the thick vertical line segment through the point $A(r, 0)$ at exactly one point $C(r, k)$, where $0 \leqslant k \leqslant r$. Afterward, the path moves from point C to point E in E- or NE-steps. That portion of the path corresponds to a unique ENE-path from the origin to the point with

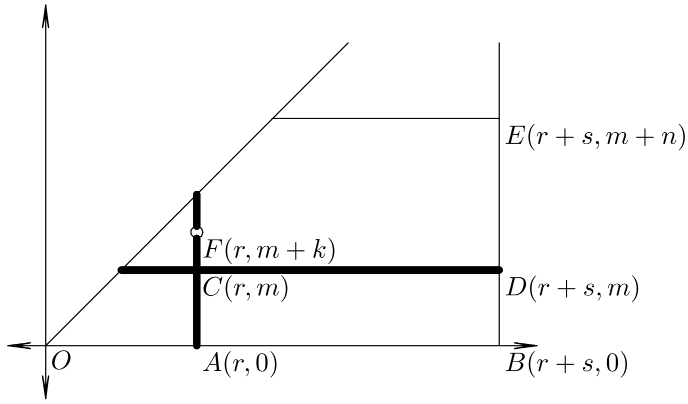

Figure 5: Second Vandermonde's convolution

coordinates $(s, n-k)$. The number of possible ENE-paths before the intersection is equal to $\binom{r}{k}$ and the number of possible continuations is equal to $\binom{s}{n-k}$. Consequently, we have the following first Vandermonde's convolution.

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}, \quad \text{integer } n. \tag{16}$$

Since there are only finitely many nonzero terms in the above sum, the polynomial argument can be applied to show that the identity holds for all reals r and s.

Suppose that in addition to a vertical reference segment we simultaneously fix a horizontal reference segment in the above counting process as illustrated in Figure 5. The same reasoning works except that the parameter k measures the displacement of the crossing point F to the horizontal reference segment instead of the displacement to the horizontal axis. We thus have the second form of Vandermonde's convolution, which also holds for all reals r and s.

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}, \quad \text{integers } m, n. \tag{17}$$

The following identity is an easy corollary of the above.

$$\sum_k \binom{p}{m+k}\binom{s}{n+k} = \binom{p+s}{p-m+n}, \quad \text{integer } p \geqslant 0, \text{ integers } m, n. \tag{18}$$

We simply replace $\binom{p}{m+k}$ by $\binom{p}{p-m-k}$, then apply (17). Since there are only finitely many nonzero terms to add up, this identity holds for all reals s by the polynomial argument.

5.2 Counting by two stages

Vandermonde's convolutions were derived by breaking down the counting of ENE-paths into two parts. This method can be further applied to the following situation. Suppose

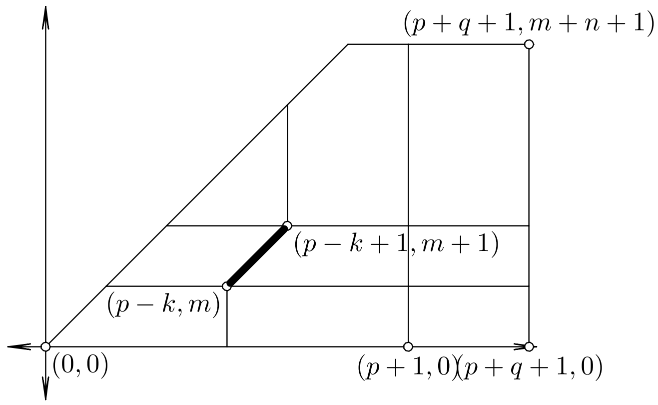

Figure 6: Diagram for identity (19)

that m, n, p, q are nonnegative integers such that $n \geqslant q$. In Figure 6, we consider ENE-paths from the origin to the point $(p+q+1, m+n+1)$. Such a path must use an NE-step to cross the band between the horizontal lines through $(0, m)$ and $(0, m+1)$. However, this NE-step cannot occur to the right of the vertical line through $(p+1, 0)$, for otherwise such a path can have at most $q-1$ NE-steps above that band, and hence it fails to reach its destination under the assumption $n \geqslant q$. Let that NE-step move from $(p-k, m)$ to $(p-k+1, m+1)$, where $0 \leqslant k \leqslant p$. It follows that such an ENE-path is determined by an ENE-path from the origin to $(p-k, m)$ and an ENE-path from $(p-k+1, m+1)$ to $(p+q+1, m+n+1)$. The latter part corresponds to a unique ENE-path from $(0,0) = (p-k+1-(p-k+1), m+1-(m+1))$ to $(p+q+1-(p-k+1), m+n+1-(m+1)) = (q+k, n)$, and vice versa. Therefore we have the following sum of products.

$$\sum_{0 \leqslant k \leqslant p} \binom{p-k}{m}\binom{q+k}{n} = \binom{p+q+1}{m+n+1}, \qquad \begin{array}{l}\text{integers } p, m \geqslant 0,\\ \text{integers } n \geqslant q \geqslant 0.\end{array} \tag{19}$$

An equivalent form can be obtained if we use $-k$ as the summing index.

$$\sum_{-p \leqslant k \leqslant 0} \binom{p+k}{m}\binom{q-k}{n} = \binom{p+q+1}{m+n+1}, \qquad \begin{array}{l}\text{integers } p, m \geqslant 0,\\ \text{integers } n \geqslant q \geqslant 0.\end{array}$$

The above method can be adapted to the reduced second quadrant. Let $p, q < 0$ and $m, n \geqslant 0$ be integers. In Figure 7, we consider all WN-paths to the point $(p+q+1, m+n+1)$. For any such WN-path, there is a unique N-step going across the band between the horizontal lines through $(0, n)$ and $(0, n+1)$. Let that N-step move from $(q+k, n)$ to $(q+k, n+1)$, where $p+q+1 \leqslant q+k \leqslant -1$. Then such a WN-path is determined by a WN-path from $(-1, 0)$ to $(q+k, n)$ and a WN-path from $(q+k, n+1)$ to $(p+q+1, m+n+1)$. The latter part corresponds to a unique WN-path from $(-1, 0) = (q+k-(q+k+1), n+1-(n+1))$ to $(p+q+1-(q+k+1), m+n+1-(n+1)) = (p-k, m)$, and vice versa.

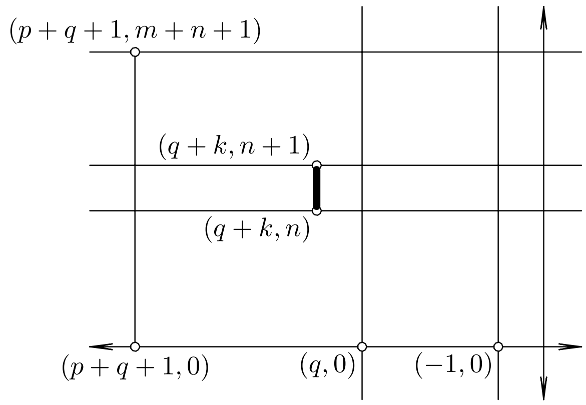

Figure 7: Diagram for identity (20)

In terms of binomial coefficients, we thus have the following sum of products in the reduced second quadrant. The minus sign on the right hand side comes from the fact $(-1)^m(-1)^n = (-1)^{m+n} = -(-1)^{m+n+1}$.

$$\sum_{p+1\leqslant k\leqslant -q-1} \binom{p-k}{m}\binom{q+k}{n} = -\binom{p+q+1}{m+n+1}, \quad \begin{array}{l}\text{integers } p,q<0,\\ \text{integers } m,n\geqslant 0.\end{array} \tag{20}$$

The following equivalent form can be obtained if we use $-k$ as the summing index.

$$\sum_{q+1\leqslant k\leqslant -p-1} \binom{p+k}{m}\binom{q-k}{n} = -\binom{p+q+1}{m+n+1}, \quad \begin{array}{l}\text{integers } p,q<0,\\ \text{integers } m,n\geqslant 0.\end{array}$$

Vandermonde's convolutions sum up products of labels on two vertical lines as the index k runs through all integers. The identities (19) and (20) give sums of products of labels on two horizontal lines within appropriate ranges.

As a matter of fact, using identities (4), (8) and (9), we can derive identities (19) and (20) from (17). The reader is encouraged to work out the details as exercises and also try his/her skills on the following identities of similar type.

Exercise 1.

$$\sum_{q+1\leqslant k\leqslant -p-1} \binom{p+k}{m}\binom{q-k}{n} = -\binom{p+q+1}{m+n+1}, \quad \begin{array}{l}\text{integers } p<0, m\geqslant 0,\\ \text{integers } n\geqslant q\geqslant 0.\end{array}$$

Exercise 2.

$$\sum_{0\leqslant k\leqslant -p-1} \binom{p+k}{m}\binom{q+k}{n} = (-1)^m\binom{q-p+m}{m+n+1}, \quad \begin{array}{l}\text{integers } p<0, m\geqslant 0,\\ \text{integers } n\geqslant q\geqslant 0.\end{array}$$

5.3 Further sums of products

Using known symmetries or signed symmetries, it is possible to transform labels on other types of lines to labels on two vertical lines so that Vandermonde's convolutions become applicable. For example, we want to compute the following sum of products.

$$S_1 = \sum_k \binom{p}{m+k}\binom{s+k}{n}(-1)^k, \quad \text{integer } p \geqslant 0, \text{ integers } m, n.$$

We first notice that each product is 0 unless $0 \leqslant m+k \leqslant p$. Now suppose that s is an integer and $s \geqslant m$. It follows that $s+k \geqslant 0$. We may first use identity (10) to obtain

$$\binom{s+k}{n} = (-1)^{s+k-n}\binom{-n-1}{s+k-n}.$$

Now the original products can be transformed into products of labels on vertical lines. We will reach the final answer by applications of identities (18), (11), and (4).

$$\begin{aligned}
S_1 &= \sum_k \binom{p}{m+k}\binom{-n-1}{s+k-n}(-1)^{s-n} \\
&= (-1)^{s-n}\sum_k \binom{p}{m+k}\binom{-n-1}{s-n+k} \\
&= (-1)^{s-n}\binom{p-n-1}{p-m+s-n} \\
&= (-1)^{p-m}\binom{s-m}{p-n+s-m} \\
&= (-1)^{p+m}\binom{s-m}{n-p}.
\end{aligned}$$

We have proved the following identity for all integers $s \geqslant m$. Hence it holds for all reals s by the polynomial argument.

$$\sum_k \binom{p}{m+k}\binom{s+k}{n}(-1)^k = (-1)^{p+m}\binom{s-m}{n-p}, \quad \text{integer } p \geqslant 0, \text{ integers } m, n. \tag{21}$$

Our final example is to compute the following sum.

$$S_2 = \sum_{k \leqslant p} \binom{p-k}{m}\binom{s}{k-n}(-1)^k, \quad \text{integers } p, m, n \geqslant 0.$$

We first use identity (10) to obtain

$$\binom{p-k}{m} = (-1)^{p-k-m}\binom{-m-1}{p-k-m}.$$

This time Vandermonde's convolution (17) applies.

$$
\begin{aligned}
S_2 &= \sum_k \binom{-m-1}{p-k-m}\binom{s}{k-n}(-1)^{p-m} \\
&= (-1)^{p+m}\sum_k \binom{s}{-n+k}\binom{-m-1}{p-m-k} \\
&= (-1)^{p+m}\binom{s-m-1}{p-m-n}.
\end{aligned}
$$

We can also conclude that the following identity holds for all reals s by the polynomial argument.

$$
\sum_{k\leqslant p}\binom{p-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{p+m}\binom{s-m-1}{p-m-n}, \quad \text{integers } p,m,n \geqslant 0. \tag{22}
$$

Remark. Identities (3), (4), (11), (13), (14), and (16) appear in Table 174 and identities (17), (18), (19), (21), and (22) appear in Table 169 of the book by Graham, Knuth, and Patashnik [1].

References

[1] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics, A foundation for computer science, Second edition*, Addison-Wesley, Reading, MA, 1994, p. 169.