NOETHERIAN GENERALIZED POWER SERIES RINGS

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ABSTRACT. Let R be a unitary ring and $(M, \leq)$ a strictly ordered monoid. We show that, if $(M, \leq)$ is positively ordered, then the generalized power series ring $R[[M, \leq]]$ is left Noetherian, if and only if, R is left Noetherian and M is finitely generated, if and only if, R is left Noetherian and $R[[M, \leq]]$ is a homomorphic image of the power series ring $R[[x_1, x_2, \dots, x_n]]$ for some $n \in \mathbb{N}$.

1. INTRODUCTION

Let R be a unitary ring, $(M, \leq)$ a strictly ordered commutative monoid and $R[[M, \leq]]$ the corresponding generalized power series ring. There are many questions one could ask about how the properties of $R[[M, \leq]]$ are determined by the properties of R and M. One of the hardest seems to be the question of when $R[[M, \leq]]$ is Noetherian. Partial answers and special cases are discussed in [6], [10] and [11]. Some examples: If $R[[M, \leq]]$ is left Noetherian, then R must be too [6, 5.2(i)]. If $R[[M, \leq]]$ is left Noetherian and M is cancellative, then M is the sum of a finitely generated monoid and an Abelian group [6, 5.2(ii)] [10]. If $(M, \leq)$ is narrow, cancellative and torsion free, M is the sum of a finitely generated monoid and an Abelian group, and R is left Noetherian, then $R[[M, \leq]]$ is Noetherian [6, 5.5] [10].

In this paper we deal with the case that $(M, \leq)$ is positively ordered. In this circumstance the partial order $\leq$ is closely tied to the algebraic structure of the monoid, and so we are able give a complete answer: $R[[M, \leq]]$ is left Noetherian if and only if R is left Noetherian and M is finitely generated.

The condition on M, that it is finitely generated, seems to say nothing about the partial order $\leq$. Indeed it turns out that the existence of a positive strict partial order on M means that, if $R[[M, \leq]]$ is left Noetherian, then $R[[M, \leq]]$ is simply the set of <u>all</u> functions from M to R, that is, the particular partial order $\leq$ does not determine $R[[M, \leq]]$. This in turn implies that $R[[M, \leq]]$ is a homomorphic image of the (usual) power series ring $R[[x_1, x_2, \dots, x_n]]$ for some $n \in \mathbb{N}$.

It is interesting to contrast the main theorem with Gilmer's theorem that, given a ring R and a monoid M, the monoid ring $R[M]$ is left Noetherian if and only if R is left Noetherian and M is finitely generated [2, 7.7] (see also [1]). Neither theorem is a special case or generalization of the other, but some of the techniques used in Gilmer's proof are also used in this paper.

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2. Ordered Sets

Let $(\mathcal{L}, \leq)$ be a partially ordered set. Then $\mathcal{L}$ is **Artinian (Noetherian)** if every nonempty subset of $\mathcal{L}$ has a minimal (maximal) element, or equivalently, $\mathcal{L}$ satisfies the descending (ascending) chain condition. An **antichain** of $\mathcal{L}$ is a subset $A \subseteq \mathcal{L}$ such that $x \leq y$ implies $x = y$ for all $x, y \in A$. $\mathcal{L}$ is **narrow** if every antichain of $\mathcal{L}$ is finite. If $\mathcal{K}$ is a subset of $\mathcal{L}$, then $\mathcal{K}$ is Artinian (Noetherian, narrow) if $\mathcal{L}$ is Artinian (Noetherian, narrow). A **lower set** of $\mathcal{L}$ is a subset $I \subseteq \mathcal{L}$ such that $x \leq y \in I$ implies $x \in I$ for all $x, y \in \mathcal{L}$. We write $\Downarrow \mathcal{L}$ for the set of lower sets of $\mathcal{L}$ ordered by inclusion.

Lemma 2.1. *Let $\sigma : \mathcal{K} \to \mathcal{L}$ be an increasing map between partially ordered sets.*

(1) *If σ is strictly increasing and $\mathcal{L}$ is Artinian (Noetherian), then $\mathcal{K}$ is Artinian (Noetherian).*
(2) *If σ is surjective and $\Downarrow \mathcal{K}$ is Artinian (Noetherian), then $\Downarrow \mathcal{L}$ is Artinian (Noetherian).*

Proof. (1) Routine.
(2) Since σ is increasing, if $I \in \Downarrow \mathcal{L}$, then $\sigma^{-1}(I) \in \Downarrow \mathcal{K}$. Since σ is surjective, σ^{-1} is a strictly increasing map from $\Downarrow \mathcal{L}$ to $\Downarrow \mathcal{K}$, and the claim follows from (1).

□

The following two lemmas are standard. See for example [3], [5], [8, 1.4].

Lemma 2.2. *Let $\mathcal{L}$ be a partially ordered set. Then the following are equivalent:*

(1) *$\Downarrow \mathcal{L}$ is Artinian.*
(2) *For every infinite sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathcal{L}$ there are $i < j$ such that $a_i \leq a_j$.*
(3) *$\mathcal{L}$ is Artinian and narrow.*

Lemma 2.3. *Let $\mathcal{K}$ and $\mathcal{L}$ be partially ordered sets. If $\Downarrow \mathcal{K}$ and $\Downarrow \mathcal{L}$ are Artinian, then $\Downarrow(\mathcal{K} \times \mathcal{L})$ is Artinian.*

Here and elsewhere in this paper, the order on a product $\mathcal{K} \times \mathcal{L}$ is defined by $(x, y) \leq (x'y')$ if $x \leq x'$ and $y \leq y'$.

3. Ordered Monoids

All monoids in this paper are commutative with monoid operation $+$. If X and Y are subsets of a monoid M, then $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$. In particular, $X + Y = \emptyset$ if $X = \emptyset$ or $Y = \emptyset$. If Y is a subset of M, we will write $\langle Y \rangle$ for the submonoid of M that Y generates.

Any monoid M has the **algebraic or natural preorder** defined by $a \preceq b$ if $a + c = b$ for some $c \in M$. In general, $a \preceq b \preceq a$ does not imply $a = b$, so $\preceq$ is not always a partial order on M. In this paper the symbol $\preceq$ will always be used for the algebraic preorder of a monoid.

A monoid F is **free** with **basis** $\emptyset \neq B \subseteq F$ if any map from B to a monoid M extends uniquely to a monoid homomorphism from F to M. In this circumstance, $F \cong \mathbb{F}^{(B)}$, the direct sum of copies of $\mathbb{F}$ indexed by B where $\mathbb{F}$ is the set of non-negative integers with addition as monoid operation. Any set B is contained as a basis in a free monoid, in particular, any monoid is a homomorphic image of a free monoid.

If a free monoid F has a finite basis B with $|B| = n \in \mathbb{N}$, then $F \cong \mathbb{F}^n \cong \mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F}$ (n times). Of course, a monoid M is finitely generated if and only if there is a surjective monoid homomorphism $\sigma : \mathbb{F}^n \to M$ for some $n \in \mathbb{N}$.

The algebraic preorder on any free monoid is a partial order. The algebraic preorder on $\mathbb{F}$ coincides with the usual order of the integers. The algebraic preorder on $\mathbb{F}^n$ is the product order. The lower sets of $(\mathbb{F}, \preceq)$ are $\emptyset$, $\mathbb{F}$ and the sets $\{0, 1, \ldots, N\}$ for $N \in \mathbb{F}$ and so $\Downarrow(\mathbb{F}, \preceq)$ is Artinian. From 2.3, $\Downarrow(\mathbb{F}^n, \preceq)$ is Artinian for $n \in \mathbb{N}$. Thus for any free monoid F with finite basis, $(F \preceq)$, and $\Downarrow(F, \preceq)$ are Artinian.

An **ordered monoid** is pair $(M, \leq)$ where M is a monoid and $\leq$ is a partial order on M such that the addition map $+ : M \times M \to M$ is increasing with respect to $\leq$, or, equivalently, $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in M$. If $(M, \leq)$ and $(N, \leq)$ are ordered monoids, then a **strict monoid homomorphism** $\sigma : (M, \leq) \to (N, \leq)$ is a monoid homomorphism $\sigma : M \to N$ which is strictly increasing with respect to the partial orders $\leq$.

A **strictly ordered monoid** is pair $(M, \leq)$ where M is a monoid and $\leq$ is a partial order on M such that the addition map $+ : M \times M \to M$ is strictly increasing with respect to $\leq$, or, equivalently, $a < b$ implies $a + c < b + c$ for all $a, b, c \in M$. If $(M, \leq)$ is an ordered monoid, then M is strictly ordered if and only if $a \leq b$ and $a + c = b + c$ imply $a = b$ for all $a, b, c \in M$.

An ordered monoid $(M, \leq)$ is **positively ordered** if $0 \leq a$ for all $a \in M$. In this circumstance, $a \preceq b$ implies $a \leq b$ for all $a, b \in M$. For example, if the algebraic order $\preceq$ on M happens to be a partial order, then $(M, \preceq)$ is a positively ordered monoid.

If an ordered monoid is both positively ordered and strictly ordered, we will say it is **positive strictly ordered** (rather than "strict positively ordered" which would conflict with our definition of a strict monoid.)

In view of the next lemma we will say that a monoid M is **strict** if $a + x + y = a$ implies $x = y = 0$ for all $a, x, y \in M$.

Lemma 3.1. *If $(M, \leq)$ is a positive strictly ordered monoid, then M is strict.*

Proof. Suppose $a + x + y = a$ for some a, x, y in M. Then $(a, 0) \leq (a, x + y)$ with $a + 0 = a + x + y$. By strictness $(a, 0) = (a, x + y)$. Thus $x + y = 0$. But then we have $(0, 0) \leq (x, y)$ with $0 + 0 = x + y$, so by strictness $(0, 0) = (x, y)$, and $x = y = 0$. □

It is easy to see that any free monoid is strict. More generally we have the following:

Lemma 3.2. *For a nontrivial monoid M the following are equivalent:*

(1) *M is strict.*
(2) *$\preceq$ is a partial order and $a + x = a$ implies $x = 0$ for all $a, x \in M$.*
(3) *There is a surjective strict monoid homomorphism $\sigma : (F, \preceq) \to (M, \preceq)$ for some free monoid F.*
(4) *$(M, \preceq)$ is a strictly ordered monoid.*

Proof. (1) $\Rightarrow$ (2): If $a \preceq b \preceq a$ in M, then there are $x, y \in M$ such that $a + x = b$ and $b + y = a$. Then $a = a + x + y$, so by strictness, $x = y = 0$ and $a = b$. Thus $\preceq$ is a partial order. The second claim is immediate.

(2) $\Rightarrow$ (3): There is certainly a monoid surjection $\sigma : F \to M$ for some free monoid F. Without loss of generality we can assume that none of the basis elements of F map to 0 under σ.

It is clear that σ is an increasing map from $(F, \preceq)$ to $(M, \preceq)$. To show it is strictly increasing it remains to show that $x \preceq y$ and $\sigma(x) = \sigma(y)$ imply $x = y$.

If $x \preceq y$ and $\sigma(x) = \sigma(y)$, then $x + z = y$ and $\sigma(x) + \sigma(z) = \sigma(y) = \sigma(x)$ for some $z \in M$. By assumption this implies $\sigma(z) = 0$. If $z \neq 0$, then there is some basis element b of F such that $z = b + w$ for some $w \in F$. But then $0 = \sigma(z) = \sigma(b) + \sigma(w)$, and we have $0 \preceq \sigma(b) \preceq 0$. Since $\preceq$ is a partial order, this implies $\sigma(b) = 0$, contrary to our choice of basis. So $z = 0$ and $x = y$.

(3) $\Rightarrow$ (4): Since $(M, \preceq)$ is at least an ordered monoid and σ is surjective, it suffices to show that, for all $a, b, c \in F$, $\sigma(a) \preceq \sigma(b)$ and $\sigma(a) + \sigma(c) = \sigma(b) + \sigma(c)$ imply $\sigma(a) = \sigma(b)$.

If $\sigma(a) \preceq \sigma(b)$, then $\sigma(b) = \sigma(a) + \sigma(d)$ for some $d \in F$, and, if $\sigma(a) + \sigma(c) = \sigma(b) + \sigma(c)$, then $\sigma(a + c) = \sigma(a + d + c)$. Since $a + c \preceq a + c + d$ and σ is strictly increasing, this implies $a + c = a + c + d$. But F is strict, so we have $d = 0$ and hence $\sigma(b) = \sigma(a) + \sigma(d) = \sigma(a)$.

(4) $\Rightarrow$ (1): This follows directly from 3.1 since $\preceq$ is a positive order on M. □

Of course, if M is a finitely generated strict monoid, then the free monoid F in (3) can be chosen to have finite basis and so there is a surjective strict monoid homomorphism $\sigma : (\mathbb{F}^n, \preceq) \to (M, \preceq)$ for some $n \in \mathbb{N}$.

Lemma 3.3. *Let M be a strict monoid. Then $\Downarrow(M, \preceq)$ is Artinian if and only if M is finitely generated.*

Proof. Suppose $\Downarrow(M, \preceq)$ is Artinian. Let $I \in \Downarrow(M, \preceq)$ be minimal such that $0 \in I$ and $M = I + \langle Y \rangle$ for some finite set $Y \subseteq M$. To prove the claim it suffices to show that $I = \{0\}$.

Suppose not. Let $0 \neq b \in I$ and set $J = I \cap \{a \in M \mid b \not\preceq a\}$. It is easy to check that J is a lower set of $(M, \preceq)$, J contains 0, and, since $b \notin J$, J is strictly smaller than I. We will show that $M = J + \langle Y, b \rangle$, which contradicts the minimality of I.

Suppose to the contrary that $M \setminus (J + \langle Y, b \rangle)$ is nonempty. Since $(M, \preceq)$ is Artinian, there is some minimal element x in $M \setminus (J + \langle Y, b \rangle)$. Since $M = I + \langle Y \rangle$, there are $i \in I$ and $y \in \langle Y \rangle$ such that $x = i + y$. The element i cannot be in J since otherwise $x \in J + \langle Y \rangle$. Thus $b \preceq i$, and $i = i' + b$ for some $i' \in I$. Set $x' = i' + y$. Then $x = x' + b$ so $x' \preceq x$, and $x' \notin J + \langle Y, b \rangle$ since otherwise the same would be true of x. By the minimality of x we get $x' = x$ and so $x' = x' + b$. Since the monoid is strict this implies $b = 0$ contrary to our assumption.

Conversely, if M is finitely generated, there is a monoid surjection $\sigma : \mathbb{F}^n \to M$ for some $n \in \mathbb{N}$. Since σ is, in particular, an increasing map from $(M, \preceq)$ to $(\mathbb{F}^n, \preceq)$, the claim follows from 2.1(2) and the fact that $\Downarrow(\mathbb{F}^n, \preceq)$ is Artinian. □

4. Generalized Power Series Rings

For a ring R and a commutative monoid M, let $R[[M]]$ be the set of all functions from M to R. Addition of such functions is defined by $(f + g)(x) = f(x) + g(x)$ for $x \in M$. $R[[M]]$ is a left R-module under the operation defined by $(rf)(x) = rf(x)$ for $x \in M$, $r \in R$ and $f \in R[[M]]$. We write $X^a \in R[[M]]$ for the function such that $X^a(a) = 1$ and $X^a(x) = 0$ if $x \neq a$. Any element $f \in R[[M]]$ can then be written in the form $f = \sum_{a \in M} f(a) X^a$.

Multiplication of $f, g \in R[[M]]$ is defined by $(fg)(x) = \sum_{x_1+x_2=x} f(x_1)g(x_2)$ for $x \in M$, if, for each $x \in M$, the set

$$\{(x_1, x_2) \mid x_1 + x_2 = x \text{ and } f(x_1) \neq 0 \text{ and } g(x_2) \neq 0\}$$

is finite. For example, $X^a X^b = X^{a+b}$ for all $a, b \in M$.

It may be that fg is defined for all $f, g \in R[[M]]$, in which case $R[[M]]$ is a ring. This happens, for example, if for each $x \in M$ the set $\{(x_1, x_2) \mid x_1 + x_2 = x\}$ is finite. Thus $R[[\mathbb{F}]]$ and more generally $R[[\mathbb{F}^n]]$, for $n \in \mathbb{N}$, are rings. These are called **rings of formal power series** with coefficients in R, and are written $R[[X]]$, and $R[[X_1, X_2, \ldots, X_n]]$ respectively.

There seems to be no proof in the literature of the following well known result except as a special case of a more general and hence more complicated theorem. See, for example [11, 4.6] or [9]. For completeness, we give a proof which is essentially that given in [4, 3.3] for the case that the ring R is commutative.

Theorem 4.1. *If R is a left Noetherian ring, then so is $R[[\mathbb{F}^n]]$.*

Proof. We prove first that $S = R[[X]] = R[[\mathbb{F}]]$ is left Noetherian, the $n = 1$ case. For a left ideal $I \leq S$ and $m \in \mathbb{F}$, let $I_m = \{f(m) \mid f \in I \cap SX^m\}$. It is easy to see that I_m is a left ideal of R and hence there is a finite set $G_m \subseteq I \cap SX^m$ such that $\{g(m) \mid g \in G_m\}$ generates I_m.

Since $I_0 \leq I_1 \leq I_2 \leq \ldots$, there is some $k \in \mathbb{F}$ such that $I_m = I_k$ for all $m \geq k$, and so, without loss of generality, we can assume that

$$G_m = X^{m-k} G_k = \{X^{m-k} g \mid g \in G_k\}$$

for $m \geq k$. Set $G = \cup_{m \leq k} G_m$, a finite set. We will show that G generates I, in particular, I is finitely generated.

Suppose $f \in I$. Since $f \in I \cap SX^0 = I$, there is a linear combination g_0 of the elements of G_0 with coefficients in R such that $f - g_0 \in I \cap SX^1$. Then there is a linear combination g_1 of the elements of G_1 with coefficients in R such that $f - g_0 - g_1 \in I \cap SX^2$. Iterating this process produces an infinite sequence $g_0, g_1, g_2, \ldots$ in I such that for $m \in \mathbb{F}$, $f - g_0 - g_1 - \ldots - g_m \in I \cap SX^{m+1}$ and g_m is an R-linear combination of the elements of G_m. Since $g_m \in SX^m$ for each m, this implies $f = \sum_{m \in \mathbb{F}} g_m$.

For $m \leq k$ we have $G_m \subseteq G$ and for $m > k$, we have $G_m \subseteq X^{m-k} G$. Hence, for each $m \in \mathbb{F}$ we can write $g_m = \sum_{g \in G} h_{mg} g$ where for $m \leq k$, $h_{mg} \in R$ and for $m > k$, $h_{mg} \in RX^{m-k}$. If we now set $h_g = \sum_{m \in \mathbb{F}} h_{mg} \in S$, then $f = \sum_{g \in G} h_g g$ as required.

The general case, $R[[\mathbb{F}^n]]$, follows by induction from the $n = 1$ case, since $R[[\mathbb{F}^n]] \cong (R[[\mathbb{F}^{n-1}]])[[\mathbb{F}]]$, as may be easily checked. □

Given a ring R and a monoid M, it may be that fg is not defined for all $f, g \in R[[M]]$. In this case one can restrict attention to certain subsets of $R[[M]]$ which do form rings. An example of this is the generalized power series ring which is constructed from a ring R and a strictly ordered monoid $(M, \leq)$ as follows: For a function $f \in R[[M]]$, define the **support** of f by $\operatorname{supp} f = \{a \in M \mid f(a) \neq 0\}$. Then the **generalized power series ring** is

$$R[[M, \leq]] = \{f \in R[[M]] \mid \Downarrow(\operatorname{supp} f, \leq) \text{ is Artinian}\},$$

that is, $R[[M,\leq]]$ is the set of functions in $R[[M]]$ whose support is Artinian and narrow in the $\leq$ partial order. See [7, 1.16] for the proof that, if $f, g \in R[[M,\leq]]$, then fg is defined and $fg \in R[[M,\leq]]$.

If $\Downarrow(M,\leq)$ happens to be Artinian, then all subsets of M are Artinian and narrow, and so $R[[M,\leq]] = R[[M]]$. For example, $\Downarrow(\mathbb{F},\preceq)$ and $\Downarrow(\mathbb{F}^n,\preceq)$ are Artinian, and so $R[[\mathbb{F},\preceq]] = R[[\mathbb{F}]]$ and $R[[\mathbb{F}^n,\preceq]] = R[[\mathbb{F}^n]]$.

The following is a special case of [7, 1.17].

Lemma 4.2. *Let R be a ring and $\sigma : (N,\leq) \to (M,\leq)$ a strict monoid homomorphism. Then σ induces a ring homomorphism $\sigma^\star : R[[N,\leq]] \to R[[M,\leq]]$. If σ is surjective, then so is $\sigma^\star$.*

Proof. Since σ is strict, $\sigma^{-1}(x)$ is an antichain in $(N,\leq)$ for all $x \in M$. Thus, if $f \in R[[N,\leq]]$, then $\sigma^{-1}(x) \cap \operatorname{supp} f$ is finite and we can define $\sigma^\star(f) = f^\star$ where $f^\star(x) = \sum_{y\in\sigma^{-1}(x)} f(y)$ for $x \in M$. It is then routine to check that $\sigma^\star$ has the required properties. □

It follows easily from the definitions that, if $f, g \in R[[M,\leq]]$, then

$$\operatorname{supp}(f+g) \subseteq \operatorname{supp} f \cup \operatorname{supp} g$$

and

$$\operatorname{supp}(fg) \subseteq \operatorname{supp} f + \operatorname{supp} g.$$

Theorem 4.3. *Let R be a ring, and $(M,\leq)$ a positive strictly ordered monoid. Then $R[[M,\leq]]$ is left Noetherian if and only if R is left Noetherian and M is finitely generated. Moreover, in this circumstance, $\Downarrow(M,\leq)$ is Artinian, and $R[[M,\leq]] = R[[M]]$ is a homomorphic image of the ring $R[[\mathbb{F}^n]]$ for some $n \in \mathbb{N}$.*

Proof. If M is trivial, then $R[[M,\leq]] \cong R$ and the claim is obvious. So we will henceforth assume that M is nontrivial. Note that, from 3.1 and 3.2, M is strict and $\preceq$ is a partial order on M.

Suppose $S = R[[M,\leq]]$ is left Noetherian.

(1) *R is left Noetherian*: [6, 5.2(i)] [10, 3.1(i)] For a left ideal $I \subseteq R$ define

$$I_S = \{f \in S \mid f(x) \in I \text{ for all } x \in M\}.$$

It is routine to check that I_S is a left ideal of S and that the map $I \mapsto I_S$ from the set of ideals of R to the set of ideals of S is strictly increasing. By 2.1(1), R is left Noetherian.

(2) *$\Downarrow(M,\preceq)$ is Artinian*: By 2.2 it suffices to show that for any infinite sequence $(a_n)_{n\in\mathbb{N}}$ in M there are $i < j$ in $\mathbb{N}$ such that $a_i \preceq a_j$.

Since S is left Noetherian and

$$SX^{a_1} \subseteq SX^{a_1} + SX^{a_2} \subseteq SX^{a_1} + SX^{a_2} + SX^{a_3} \subseteq \dots,$$

there is some $j \in \mathbb{N}$ such that $X^{a_j} \in \sum_{i<j} SX^{a_i}$. Thus $X^{a_j} = \sum_{i<j} f_i X^{a_i}$ for some $f_1, f_2, \ldots \in S$ and $a_j \in \cup_{i<j}(a_i + \operatorname{supp} f_i)$. This means that for some $i < j$ and $t \in \operatorname{supp} f_i$ we have $a_j = a_i + t$, in particular, $a_i \preceq a_j$.

(3) *M is finitely generated*: Since M is strict, this follows immediately from (2) and 3.3.

Conversely, suppose that R is left Noetherian and M is finitely generated. From 3.2, there is a strict monoid surjection $\sigma : (\mathbb{F}^n,\preceq) \to (M,\preceq)$ for some $n \in \mathbb{N}$. Since $\leq$ is a positive order, we have $a \preceq b \implies a \leq b$ for all $a, b \in M$. In other words, the identity map from $(M, \preceq)$ to $(M, \leq)$ is a strict monoid surjection. Composing these two maps gives a strict monoid surjection from $(\mathbb{F}^n, \preceq)$ to $(M, \leq)$, and so by 4.2, $R[[M, \leq]]$ is a homomorphic image of the ring $R[[\mathbb{F}^n, \preceq]] = R[[\mathbb{F}^n]]$. Since $R[[\mathbb{F}^n]]$ is left Noetherian, so is $R[[M, \leq]]$.

Finally, if $R[[M, \leq]]$ is left Noetherian, then from above, $\Downarrow(M, \preceq)$ is Artinian. Applying 2.1(2) to the identity map $(M, \preceq)$ to $(M, \leq)$ we get that $\Downarrow(M, \leq)$ is Artinian. Thus all subsets of $(M, \leq)$ are Artinian and narrow and $R[[M, \leq]] = R[[M]]$. □

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