California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Applied Probability Theory   Fall 2017
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Do five of the following seven problems as indicated below. You do not need to simplify algebraic expressions in your final answers. All problems are worth the same number of points. To receive full credit, show all your work and give reasons for your answers.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2017 # 1. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn, c additional balls of the same color are put in with it. Now suppose that we draw another ball at random. What is the probability that the first ball drawn was black given that the second ball drawn was red?

Fall 2017 # 2. Let X and Y be independent random variables both uniformly distributed on $(0, a)$.
(a) Find the probability density function of the random variable $X + Y$.
(b) Find $\text{Var}(X + Y)$.

Fall 2017 # 3. The joint probability density function of random variables X and Y is given by

$$f(x, y) = \begin{cases} c(x+y) & \text{for } x > 0,\ y > 0, \text{ and } x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

(a) Find the conditional p.d.f. of X given that $Y = y$.
(b) Find $E(X \mid Y = y)$ and $\text{Var}(X \mid Y = y)$.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2017 # 4. Suppose that the conditional distribution of N, given that $Y = y$, is Poisson with mean y. Further suppose that Y is a gamma random variable with parameters (r, λ), where r is a positive integer. That is,

$$P(N = n | Y = y) = \frac{e^{-y} y^n}{n!}, \text{ and } \quad f_Y(y) = \frac{\lambda e^{-\lambda y} (\lambda y)^{r-1}}{(r-1)!} \quad \text{for } y > 0.$$

(a) Find $E(N)$.
(b) Find Var(N).

Fall 2017 # 5. A Markov chain $\{X_n, n \geq 0\}$ with states 0, 1, 2, has the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(a) Specify the classes of the Markov chain, and determine whether they are transient or recurrent.
(b) If $P(X_0 = 0) = 1/2$ and $P(X_0 = 1) = 1/3$, find $E(X_3)$.

Fall 2017 # 6. (a) Let X_1, X_2, and X_3 be independent exponential random variables with parameters (rates) μ_i, i = 1, 2, 3.

(1) Determine the probability distributions of the random variables $Y = \min\{X_2, X_3\}$ and $Z = \min\{X_1, X_2, X_3\}$.
(2) Find $P(X_1 < Y)$.

(b) Customers (waiting in a single line) can be served by any of three servers, where the service times of server i are exponentially distributed with parameter (rate) μ_i, i = 1, 2, 3. Whenever a server becomes free, the customer who has been waiting the longest begins service with that server. If you arrive to find all three servers busy and no one waiting, find the expected time until you depart the system.

Fall 2017 # 7. In a bicycle race between two competitors, Let $X(t)$ denote the amount of time (in seconds) by which the racer that started in the inside position is ahead when $100t$ percent of the race has been completed, and suppose that $\{X(t), 0 \leq t \leq 1\}$ can be modeled as a Brownian motion process with variance parameter σ^2.

(a) What is the distribution of $X(1/3) + X(1/2)$?
(b) If the inside racer is leading by 2σ seconds at the midpoint of the race, what is the probability that she is the winner? Justify your answer.