

California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Applied Probability Theory Fall 2018
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Do five of the following seven problems as indicated below. You do not need to simplify algebraic expressions in your final answers. All problems are worth the same number of points.

To receive full credit, make sure to give reasons for your answers, and to clearly mark your answers.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Fall 2018 # 1. There are two doors. Behind each door there is a magical suitcase, which either contains \$100,000 cash or a snake, each with probability $1/2$.

(a) Someone randomly chooses one of the doors and opens the suitcase behind the door. What is the probability that both suitcases contain \$100,000 cash if the revealed suitcase contains \$100,000 cash? Give reasons for your answer and carefully define the variables used in your arguments.

(b) Instead of choosing randomly, the person first peeks into both suitcases, then opens one of them using the rule that if either of the suitcases contains \$100,000 cash, then she always chooses to reveal the one with cash. What is the probability that both suitcases contain cash if the revealed one contains cash?

Make sure to carefully define your variables and give reasons for your answers.

Fall 2018 # 2. Suppose two random variables X and Y have a joint probability density function

$$f(x, y) = 2e^{(x/2)-4y} \quad \text{for } 0 < x, y < \infty; \quad f(x, y) = 0 \quad \text{otherwise.}$$

Let $Z = \max\{X, Y\}$.

(a) Find the marginal probability density functions of X and Y , respectively. Are X and Y independent?

(b) Find the probability density function of Z .

(c) Find the expectation $E(Z)$.

Fall 2018 # 3. Suppose two random variables X and Y have a joint probability density function

$$f(x, y) = x + y \quad \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1; \quad f(x, y) = 0 \quad \text{otherwise.}$$

(a) Find the conditional probability density function of X given $Y = y$.

(b) Find the conditional expectation $E(X|Y = y)$.

(c) Find the covariance $\text{Cov}(X, Y)$.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2018 # 4. A set of n dice is thrown. All those that land on six are put aside, and the others are again thrown. This is repeated until all the dice have landed on six. Let N denote the number of throws needed. (For instance, suppose that $n = 3$ and that on the initial throw exactly two of the dice land on six. Then the other die will be thrown, and if it lands on six, then $N = 2$.) Let $m_n = E[N]$.

(a) Show that $m_1 = 6$ and $m_2 = 96/11$. (Hint: Condition on number of sixes on first throw).

(b) Derive a recursive formula for m_n .

Make sure you explain your reasoning.

Fall 2018 # 5. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red ball, 1 white ball, and 2 blue balls; the white urn contains 2 white balls and 1 blue ball; the blue urn contains 3 white balls and 1 blue ball. At the initial stage ($n = 0$), a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. A Markov chain $\{X_n, n = 0, 1, 2, \dots\}$ can be used to analyze this system, with the three states 0, 1, 2, where 0 = red, 1 = white, and 2 = blue indicate the color of the ball drawn.

(a) Find the transition probability matrix of the Markov chain.

(b) Specify the classes of the Markov chain and determine whether the classes are transient or recurrent. Justify your answer.

(c) Express $E(X_2)$ in terms of elements of an appropriate power of the transition matrix. You do not have to compute these matrix elements.

(d) In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?

Fall 2018 # 6. (a) Let X_1 and X_2 be independent exponential random variables with parameters (rates) $\mu_i, i = 1, 2$. Find the conditional probabilities $P(X_1 > 2X_2)$ and $P(X_1 > 2X_2 | X_1 > X_2)$.

(b) A post office has three servers. Each entering customer must be served first by server 1, then by server 2, and finally by server 3. The amount of time it takes to be served by server i is an exponential random variable with rate $\mu_i, i = 1, 2, 3$. Suppose you enter the system when it contains a single customer who is being served by server 3.

(1) Find the probability that server 3 will still be busy when you move over to server 2.

(2) Find the probability that server 3 will still be busy when you move over to server 3.

(3) Find the expected amount of time that you spend in the system. (Whenever you encounter a busy server, you must wait for the server in progress to end before you can enter service.)

Fall 2018 # 7. (a) Let $\{B(t), t \geq 0\}$ be a standard Brownian motion process.

(1) What is the distribution of $B(s) + 2B(t)$, where $s < t$.

(2) Find the value of the probability $B(5) > 0$ given that $B(1) = 2$.

(b) Let $\{X(t), t \geq 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2 .

(1) What is the joint probability density function of $X(s)$ and $X(t)$, where $s < t$?

(2) What is the conditional distribution of $X(t)$ given that $X(s) = c$, where $s < t$?