

California State University – Los Angeles  
Department of Mathematics  
Master's Degree Comprehensive Examination  
Applied Probability Theory      Spring 2017  
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Do five of the following seven problems as indicated below. You do not need to simplify algebraic expressions in your final answers. All problems are worth the same number of points.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

**SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.**

**Spring 2017 # 1.** We have two bags each containing red and blue balls. Bag A contains 10 red and 15 blue balls, and bag B contains 4 red and 9 blue balls. Suppose that we choose one of the bags at random. For this bag we select 3 balls at random without replacement. The result is that we find one red and two blue balls. What is the probability that we chose the bag B?

**Spring 2017 # 2.** Let  $X_1, X_2, \dots, X_n$  be a random sample from the exponential distribution with p.d.f.

$$f(x) = \beta e^{-\beta x} \text{ for } x > 0; \quad f(x) = 0 \text{ for } x \leq 0.$$

Let  $Y_n = \max\{X_1, X_2, \dots, X_n\}$ . (a) Find the cumulative distribution function (c.d.f.) and the probability density function (p.d.f.) of  $Y_n$ . (b) Find the expectation  $E(Y_n)$ .

**Spring 2017 # 3.** The joint probability density function of random variables  $X$  and  $Y$  is given by

$$f(x, y) = \frac{e^{-x/y} e^{-y}}{y}, \quad 0 < x < \infty, \quad 0 < y < \infty.$$

(a) Find the conditional p.d.f. of  $X$  given that  $Y = y$ . (b) Show that  $E(X|Y = y) = y$ .

**SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.**

**Spring 2017 # 4.** A coin having probability  $p$  of landing heads is continually flipped until at least one head and one tail have been flipped. (a) Find the expected number of flips needed. (b) Find the expected number of flips that land on heads. (Hint: You may compute the expected values by conditioning)

**Spring 2017 # 5.** (a) A professor continually gives exams to her students. She can give three possible types of exams, as her class is graded as either having done well or badly. Let  $p_i$  denote the probability that the class does well on a type  $i$  exam, and suppose that  $p_1 = 0.3$ ,  $p_2 = 0.6$ , and  $p_3 = 0.9$ . If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does

badly, then the next exam is always type 1. (1) Construct a 3-by-3 transition probability matrix. (2) What proportion of exams are type  $i$ ,  $i = 1, 2, 3$  (in the long run)?

(b) Consider a Markov chain with states equal to the nonnegative integers, and suppose its transition probabilities satisfy  $P_{i,j} = 0$  if  $j \leq i$ . Assume  $X_0 = 0$ , and let  $e_j$  be the probability that the Markov chain is ever in state  $j$ . (Note that  $e_0 = 1$  because  $X_0 = 0$ ). Argue that for  $j > 0$ ,

$$e_j = \sum_{i=0}^{j-1} e_i P_{i,j}$$

If  $P_{i,i+k} = 1/3$ ,  $k = 1, 2, 3$ , find  $e_1$ ,  $e_2$ , and  $e_3$ .

**Spring 2017 # 6.** Consider a two-server system in which a customer is served first by server 1, then by server 2, and then departs. The service times at server  $i$  are exponential random variables with parameters (rates)  $\beta_i$ ,  $i = 1, 2$ . When you arrive, you find server 1 free and two customers at server 2 – customer  $A$  in service and customer  $B$  waiting in line.

(a) Find  $P_A$ , the probability that  $A$  is still in service when you move over to server 2.

(b) Find  $P_B$ , the probability that  $B$  is still in the system when you move over to server 2. (Hint: You can

use the formula  $\int_0^x \int_0^{x-y} \beta_2 e^{-\beta_2 y} \beta_2 e^{-\beta_2 z} dz dy = 1 - (e^{-\beta_2 x} + \beta_2 x e^{-\beta_2 x})$ .)

**Spring 2017 # 7.** Let  $\{B(t), t \geq 0\}$  be a standard Brownian motion process, and let  $T_a$  denote the time it takes to hit  $a$ . (a) What is the distribution of  $B(s) + B(t)$ ,  $s < t$ ? (b) What is  $P(T_1 < T_{-2} < T_5)$ ? Justify your answer.