California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Applied Probability Theory Spring 2018
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Do five of the following seven problems as indicated below. You do not need to simplify algebraic expressions in your final answers unless specifically asked to do so. All problems are worth the same number of points.

To receive full credit, make sure to give reasons for your answers, and to clearly mark your answers.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2018 # 1. Suppose that 20 percent of the bottles produced in a certain plant are defective. If a bottle is defective, the probability that an inspector will notice it and remove it from the filling line is 0.9. If a bottle is not defective, the probability that the inspector will think that it is defective and remove it from the filling line is 0.2.
(a) If a bottle is removed from the filling line, what is the probability that it is not defective?
(b) If two bottles are removed from the filling line, what is the probability that both are defective?
Make sure to carefully define the relevant events and to justify your answers.

Sping 2018 # 2. Let X and Y have joint probability density function

$$f(x, y) = cx(y-x)e^{-y} \quad \text{for } 0 \le x \le y < \infty.$$

(a) Find c.
(b) Show that

$$f_{X|Y}(x|y) = 6x(y-x)y^{-3} \quad \text{for } 0 \le x \le y < \infty,$$

$$f_{Y|X}(y|x) = (y-x)e^{x-y} \quad \text{for } 0 \le x \le y < \infty.$$

(c) Deduce that $E(X|Y) = \frac{1}{2}Y$ and $E(Y|X) = X + 2$.

You may use the formula $\int x^n e^{-x} dx = -e^{-x} \sum_{k=0}^{n} \frac{n!}{(n-k)!} x^{n-k}$.

Sping 2018 # 3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on [0, 1], and let $Y_1 = \min\{X_1, X_2, \ldots, X_n\}$ and $Y_n = \max\{X_1, X_2, \ldots, X_n\}$.

(a) Find the cumulative distribution function (c.d.f.) and the probability density function (p.d.f.) of Y_n.

(b) Find the expectation $E(Y_n)$.

(c) Find the c.d.f. and the p.d.f. of Y_1.

(d) Find the expectation $E(Y_1)$.

(e) Find $E(Y_n - Y_1)$, the expectation of the range of the random sample.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Spring 2018 # 4. A fair coin is tossed successively. Let K_n be the number of tosses until n consecutive heads occur.

(a) Derive an expression for the conditional expectation $E(K_n | K_{n-1} = i)$ and use it to show that

$$E(K_n | K_{n-1}) = K_{n-1} + 1 + \frac{1}{2} E(K_n).$$

(b) Use part (a) to find a recursive relation between $E(K_n)$ and $E(K_{n-1})$. (Hint: Use conditioning.)

(c) Compute $E(K_1)$. (Hint: Geometric distribution)

(d) Use the recursion in part (b) and the result of part (c) to find an explicit expression for $E(K_n)$.

Spring 2018 # 5. Consider a system consisting of two devices (working independently of each other) that can both take the values *broken* and *functioning*. Every day the individual devices break down with probabilities p_1 and p_2, respectively. The system is inspected every morning and broken devices (if any) are replaced by the following morning. The state of the system on the morning of day n can be described by a Markov chain with the four states (0, 0), (0, 1), (1, 0), (1,1), where 0 = *broken* and 1 = *functioning*. Let $X(n)$ denote the state of the system on day n and assume that no device is broken on the morning of day $n = 0$.

(a) Find the possible transitions of the four state Markov chain and draw the transition diagram of the chain without transition probabilities.

(b) Find the transition matrix of this Markov chain.

(c) Compute the distribution of $X(1)$, i.e., find $P(X(1) = (i, j))$, with $i, j = 0, 1$.

(d) Let $\pi_{ij} = \lim_{n\to\infty} P(X(n) = (i, j))$, $i, j = 0, 1$ be the limiting distribution of $X(n)$. Show that the solution to the system of equations for the probabilities $\pi_{0,0}$, $\pi_{0,1}$, $\pi_{1,0}$, and $\pi_{1,1}$ is given by

$$\pi_{ij} = \frac{p_1^{1-i} p_2^{1-j}}{(1+p_1)(1+p_2)}, \quad i, j = 0, 1.$$

Spring 2018 # 6.

(a) Let X_1 and X_2 be independent exponential random variables with parameters (rates) μ_i, $i = 1, 2$.

(1) Determine the probability density function of the random variable $Y = \max\{X_1, X_2\}$.

(2) Find $E(Y)$.

(b) Customers can be served by any of two servers in a post office, where the service time of server i is exponentially distributed with parameter (rate) μ_i, $i = 1, 2$. Suppose that when customer A enters the system he discovers that customer B is being served by one server and customer C by the other. Suppose that customer A is told that his service will begin as soon as either customer B or customer C leaves. What is the expected time until all three customers have left the post office?

Spring 2018 # 7.

(a) Let $\{B(t), t \geq 0\}$ be a standard Brownian motion process.

(1) What is the distribution of $2B(s) + 3B(t)$ when $s < t$?

(2) What is the probability $P(B(6) > 1 \mid B(2) = 3)$?

(b) Let $\{X(t), t \geq 0\}$ be a Brownian motion with drift coefficient μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given that $X(s) = c$ when $s < t$?

(c) Let Z be a standard Normal random variable and let $X(t) = \sqrt{t}Z$ for $t \geq 0$. Is $\{X(t), t \geq 0\}$ a Brownian motion? Give reasons for your answer.