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Math 540A

# CHAPTER 4 HOMEWORK

## Sec 4.3 # 2a, 4, 8, 11ab

2.) Find all Conjugacy Classes & their sizes in $D_8$.

$$D_8 = \{\cancel{1}, \cancel{r}, \cancel{r^2}, \cancel{r^3}, \cancel{s}, sr, \cancel{sr^2}, sr^3\}$$

$$1, r^2 \in Z(D_8)$$

$$\begin{aligned} D_8 \cdot r &= \{1\cdot r, r\cdot r, r^2\cdot r, r^3\cdot r, s\cdot r, sr\cdot r, sr^2\cdot r, sr^3\cdot r\} \\ &= \{r, r, r, r, srs^{-1}, (sr)r(sr)^{-1}, (sr^2)r(sr^2)^{-1}, (sr^3)r(sr^3)^{-1}\} \\ &= \{r, r, r, r, r^3, r^3, r^3, r^3\} \\ &= \{r, r^3\} = D_8 \cdot r^3 \end{aligned}$$

$$\begin{aligned} D_8 \cdot s &= \{1\cdot s, r\cdot s, r^2\cdot s, r^3\cdot s, s\cdot s, sr\cdot s, sr^2\cdot s, sr^3\cdot s\} \\ &= \{s, rsr^{-1}, r^2sr^{-2}, r^3sr^{-3}, sss^{-1}, (sr)s(sr)^{-1}, (sr^2)s(sr^2)^{-1}, (sr^3)s(sr^3)^{-1}\} \\ &= \{s, sr^2, s, sr^2, s, sr^2, s, sr^2\} \\ &= \{s, sr^2\} = D_8 \cdot sr^2 \end{aligned}$$

$$\begin{aligned} D_8 \cdot sr &= \{1\cdot sr, r\cdot sr, r^2\cdot sr, r^3\cdot sr, s\cdot sr, sr\cdot sr, sr^2\cdot sr, sr^3\cdot sr\} \\ &= \{sr, rsrr^{-1}, r^2srr^{-2}, r^3srr^{-3}, ssrs^{-1}, (sr)(sr)(sr)^{-1}, (sr^2)sr(sr^2)^{-1}, (sr^3)sr(sr^3)^{-1}\} \\ &= \{sr, sr^3, sr, sr^3, sr^3, sr, sr^3, sr\} \\ &= \{sr, sr^3\} = D_8 \cdot sr^3 \end{aligned}$$

| Conjugacy Class | Size |
|---|---|
| $D_8 \cdot 1 = \{1\}$ | 1 |
| $D_8 \cdot r^2 = \{r^2\}$ | 1 |
| $D_8 \cdot r = \{r, r^3\}$ | 2 |
| $D_8 \cdot s = \{s, sr^2\}$ | 2 |
| $D_8 \cdot sr = \{sr, sr^3\}$ | 2 |

4.) Prove: If $S \subseteq G$ & $g \in G$, then...

a.) $gN_G(S)g^{-1} = N_G(gSg^{-1})$:

Pf: $N_G(S) = \{x \in G : xSx^{-1} = S\}$

$\Rightarrow gN_G(S)g^{-1} = \{gxg^{-1} : xSx^{-1} = S\}$

$= \{gxg^{-1} : g(xSx^{-1})g^{-1} = gSg^{-1}\}$

Let $u = gxg^{-1}$

$\Rightarrow gx = ug \ \wedge \ x^{-1}g^{-1} = g^{-1}u^{-1}$

$\Rightarrow gN_G(S)g^{-1} = \{u \in G : u(gSg^{-1})u^{-1} = gSg^{-1}\} = N_G(gSg^{-1})$ ▨

b.) $gC_G(S)g^{-1} = C_G(gSg^{-1})$:

Pf: $C_G(S) = \{x \in G : xsx^{-1} = s \ \forall s \in S\}$

$\Rightarrow gC_G(S)g^{-1} = \{gxg^{-1} : xsx^{-1} = s \ \forall s \in S\}$

$= \{gxg^{-1} : g(xsx^{-1})g^{-1} = gsg^{-1} \ \forall s \in S\}$

Let $u = gxg^{-1}$

$\Rightarrow gx = ug \ \wedge \ x^{-1}g^{-1} = g^{-1}u^{-1}$

$\Rightarrow gC_G(S)g^{-1} = \{u \in G : u(gsg^{-1})u^{-1} = gsg^{-1} \ \forall s \in S\} = C_G(gSg^{-1})$ ▨

8.) Prove: $Z(S_n) = \{1\}$ if $n \geq 3$

Pf: Let $\sigma \in S_n \setminus \{1\}$, where $n \geq 3$. Let $A = \{1, 2, \ldots, n\}$

$\Rightarrow \exists \ a, b \in A$ s.t. $\sigma(a) = b$, where $a \neq b$

$\wedge \ \exists \ c, d \in A$ s.t. $\sigma(c) = d$, where $a \neq c$ & $c \neq d$

$\Rightarrow b \neq d$, since $\sigma$ is 1-1

Let $\tau = (b, d) \in S_n$

$$\Rightarrow \begin{cases} (\sigma \circ \tau)(a) = \sigma(\tau(a)) = \sigma(a) = b \\ (\tau \circ \sigma)(a) = \tau(\sigma(a)) = \tau(b) = d \end{cases}$$

$\Rightarrow \sigma \circ \tau \neq \tau \circ \sigma$

$\Rightarrow \sigma \notin Z(S_n)$

Since $1 \in Z(S_n)$, we know $Z(S_n) = \{1\}$ ▨

11.) Determine whether $\sigma_1$ & $\sigma_2$ are conjugate. If so, find a $\tau \in S_n$ s.t. $\tau\sigma_1\tau^{-1} = \sigma_2$

a.) $\sigma_1 = (1,2)(3,4,5)$

$\quad\quad\;\; \downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow$

$\sigma_2 = (4,5)(1,2,3)$

$\sigma_1$ & $\sigma_2$ are conjugate. (Both have Cycle Type 2,3)

$\tau(1)=4 \quad \tau(2)=5 \quad \tau(3)=1 \quad \tau(4)=2 \quad \tau(5)=3$

$\therefore \tau = (1,4,2,5,3)$

b.) $\sigma_1 = (1,5)(3,7,2)(10,6,8,11)(4)(9)(12)(13)$

$\quad\quad\;\; \downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow\;\downarrow$

$\sigma_2 = (4,9)(13,11,2)(3,7,5,10)(1)(6)(8)(12)$

$\sigma_1$ & $\sigma_2$ are conjugate. (Both have Cycle Type 1,1,1,1,2,3,4)

$\tau(1)=4 \quad \tau(2)=2 \quad \tau(3)=13 \quad \tau(4)=1 \quad \tau(5)=9$

$\tau(6)=7 \quad \tau(7)=11 \quad \tau(8)=5 \quad \tau(9)=6 \quad \tau(10)=3$

$\tau(11)=10 \quad \tau(12)=8 \quad \tau(13)=12$

$\therefore \tau = (1,4)(3,13,12,8,5,9,6,7,11,10)$

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# Sec 4.4 # 2, 15

2.) Prove: If G is Abelian of size pq f.s. distinct primes p & q, then G is Cyclic.

Pf: Let G be Abelian. Let $|G| = pq$ f.s. distinct primes p & q.
$\Rightarrow \exists\, x, y \in G$ s.t. $|x| = p \wedge |y| = q$ by Cauchy's Thm
$\Rightarrow x \neq 1_G \wedge y \neq 1_G$
Let $|xy| = n$. Since G is Abelian, we know...
$(xy)^{pq} = x^{pq} y^{pq} = (x^p)^q (y^q)^p = 1_G 1_G = 1_G$
$\Rightarrow n \mid pq$
$\Rightarrow n = 1, p, q,$ or $pq$
If $n = 1$, then $xy = 1_G$
$\Rightarrow x = y^{-1}$
$\Rightarrow 1_G = x^p = y^{-p}$, a Contradiction (since $q \nmid p$)
If $n = p$, then $(xy)^p = 1_G$
$\Rightarrow 1_G = x^p y^p = 1_G y^p = y^p$, a Contradiction (since $q \nmid p$)
If $n = q$, then $(xy)^q = 1_G$
$\Rightarrow 1_G = x^q y^q = x^q 1_G = x^q$, a Contradiction (since $p \nmid q$)
So $|xy| = n = pq = |G|$
$\therefore G = \langle xy \rangle$ is Cyclic ▨

15.) Prove: $\mathbb{Z}_5^\times$, $\mathbb{Z}_9^\times$, & $\mathbb{Z}_{18}^\times$ are Cyclic.

a.) $\mathbb{Z}_5^\times \cong Aut(\mathbb{Z}_5) \cong \mathbb{Z}_4$ is Cyclic, since 5 is an odd prime.

b.) $|\mathbb{Z}_9^\times| = |\{\bar{a} \in \mathbb{Z}_9 : \gcd(a, 9) = 1\}| = |\{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}| = 6 = 2 \cdot 3$
$\Rightarrow \mathbb{Z}_9^\times$ is Cyclic.

c.) $|\mathbb{Z}_{18}^\times| = |\{\bar{a} \in \mathbb{Z}_{18} : \gcd(a, 18) = 1\}| = |\{\bar{1}, \bar{5}, \bar{7}, \overline{11}, \overline{13}, \overline{17}\}| = 6 = 2 \cdot 3$
$\Rightarrow \mathbb{Z}_{18}^\times$ is Cyclic.

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Sec 4.5 # 5, 13, 30

5.) Prove: A Sylow p-Subgroup of $D_{2n}$ is Cyclic & Normal $\forall$ odd prime p.

Pf: Let p be an odd prime. Let $|D_{2n}| = 2n = p^\alpha m$, where $p \nmid m$. Since p is odd, we know m is even. Let $m=2k$, so $p^\alpha = \frac{2n}{m} = \frac{2n}{2k} = \frac{n}{k}$. Let $H = \langle r^k \rangle$.

$\Rightarrow |H| = |r^k| = \frac{n}{\gcd(k,n)} = \frac{n}{k} = p^\alpha$ (so $H \in Syl_p(D_{2n})$)

CLAIM: $H \trianglelefteq D_{2n}$

Pf: $H = \langle r^k \rangle = \{(r^k)^h : h \in \mathbb{Z}\}$. Let $a \in \mathbb{Z}$

i.) $r^a H (r^a)^{-1} = \{r^a (r^k)^h r^{-a} : h \in \mathbb{Z}\} = \{r^{a+kh-a} : h \in \mathbb{Z}\} = \{(r^k)^h : h \in \mathbb{Z}\} = H$

ii.) $(sr^a) H (sr^a)^{-1} = \{(sr^a) r^{kh} (sr^a)^{-1} : h \in \mathbb{Z}\} = \{sr^{a+kh} sr^a : h \in \mathbb{Z}\} = \{s^2 r^{-a-kh+a} : h \in \mathbb{Z}\}$
$= \{(r^{-k})^h : h \in \mathbb{Z}\} = \langle r^{-k} \rangle = \langle r^k \rangle = H$

$\therefore H = \langle r^k \rangle \trianglelefteq D_{2n}$ is a Cyclic Normal Sylow p-Subgroup of $D_{2n}$ ▨

13.) Prove: if $|G| = 56$, then G has a Normal Sylow p-Subgroup f.s. prime dividing 56.

Pf: Assume $|G| = 56 = 2^3 \cdot 7$

$\Rightarrow n_7 \equiv 1 \pmod 7 \wedge n_7 | 8$ by Sylow 3

$\Rightarrow n_7 = 1$ or $8$ (since $2, 4 \not\equiv 1 \pmod 7$)

CASE 1: $n_7 = 1$

$\Rightarrow \exists!\, P \in Syl_7(G)$

$\Rightarrow gPg^{-1} = P \;\forall g \in G$, so $P \trianglelefteq G$

CASE 2: $n_7 = 8$. Let $Syl_7(G) = \{P_1, P_2, \ldots, P_8\}$, where $P_i \neq P_j$ if $i \neq j$.

CLAIM: $P_i \cap P_j = \{1_G\}$ if $i \neq j$

Pf: Let $x \in P_i \cap P_j$, where $i \neq j$

$\Rightarrow |x| = 1$ or $7$ by Lagrange

But if $|x| = 7$, then $P_i = \langle x \rangle = P_j$, a Contradiction. So $|x| = 1$.

$\Rightarrow x = 1_G$, so $P_i \cap P_j = \{1_G\}$

By this Claim, each order-7 element is within EXACTLY one Sylow 7-Subgroup of G. By Lagrange, $P_i$ has 6 elements of order 7 $\forall 1 \leq i \leq 8$.

$\therefore$ G has EXACTLY $8 \cdot 6 = 48$ elements of order 7.

By Sylow 3, $n_2 \equiv 1 \pmod 2 \wedge n_2 | 7$

$\Rightarrow n_2 = 1$ or $7$

But only $56 - 48 = 8$ elements of G do not have order 7. Any Sylow 2-Subgroup of G has $2^3 = 8$ elements, so $n_2 \neq 7$

$\Rightarrow n_2 = 1$

$\Rightarrow \exists!\, Q \in Syl_2(G)$

$\Rightarrow gQg^{-1} = Q \;\forall g \in G$, so $Q \trianglelefteq G$

$\therefore$ G has a Normal Sylow 7-Subgroup OR a Normal Sylow 2-Subgroup. ▨

30.) How many order-7 elements must there be in a Simple Group of order 168?

ANSWER: 48

Pf: Let G be a Simple Group s.t. $|G| = 168 = 2^3 \cdot 3 \cdot 7$.

$\Rightarrow n_7 \equiv 1 \pmod 7 \wedge n_7 | 24$ by Sylow 3

$\Rightarrow n_7 = 1$ or $8$ (since $2, 3, 4, 6, 12, 24 \not\equiv 1 \pmod 7$)

But G is Simple, so $\nexists$ a Normal Sylow 7-Subgroup of G.

$\Rightarrow n_7 \neq 1$

$\Rightarrow n_7 = 8$

By the Claim in #13, each order-7 element is within EXACTLY one Sylow 7-Subgroup of G. By Lagrange, each Sylow 7-Subgroup of G has 6 elements of order 7.

$\therefore$ G contains EXACTLY $8 \cdot 6 = 48$ elements of order 7. ▨