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Math 540A

CHAPTERS 5 & 6 HOMEWORK

Sec 5.2 # A, B

A.) Let G be a finite Abelian group. Prove G is Simple iff $G \cong \mathbb{Z}_p$ f.s. prime p.

Pf: Suppose G is Abelian s.t. $|G| = n$ f.s. $n \in \mathbb{Z}^+$. Since G is Abelian, ALL subgroups of G are Normal.

($\Leftarrow$): Assume $G \cong \mathbb{Z}_p$ f.s. prime p. Let $H \leq G$
$\Rightarrow |H| = 1$ or p by Lagrange
$\Rightarrow H = \{1_G\}$ or G
$\Rightarrow$ G is Simple.

($\Rightarrow$): Assume $G \not\cong \mathbb{Z}_p$ for any prime.
Case 1: G is cyclic. Let $G = \langle x \rangle = \{1_G, x, x^2, \ldots, x^{n-1}\}$, where $|G| = |x| = n$ is not prime.
$\Rightarrow n = ab$ f.s. $1 < a \leq b < n$
$\Rightarrow \langle x^a \rangle = \{1_G, x^a, x^{2a}, \ldots, x^{(b-1)a}\}$, since $x^{ba} = x^n = 1_G$
$\Rightarrow 1 < b = |\langle x^a \rangle| < n$
$\Rightarrow \langle x^a \rangle \neq \{1_G\} \wedge \langle x^a \rangle \neq G$
$\Rightarrow$ G is not Simple.
Case 2: G is NOT cyclic. But G is finite Abelian, so $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_s}$, where $|G| = n_1 n_2 \ldots n_s$ & $n_i \geq 2\ \forall i$.
Let $H = \{\bar{0}\} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_s}$.
CLAIM: $H \leq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_s}$
Pf: We use Subgroup Criterion:
a.) $(\bar{0}, \bar{0}, \ldots, \bar{0}) \in H$, so $H \neq \emptyset$
b.) Let $(\bar{0}, \bar{a_2}, \ldots, \bar{a_s}), (\bar{0}, \bar{b_2}, \ldots, \bar{b_s}) \in H$
$\Rightarrow (\bar{0}, \overline{-b_2}, \ldots, \overline{-b_s}) \in H$
$\Rightarrow (\bar{0}, \bar{a_2}, \ldots, \bar{a_s}) + (\bar{0}, \bar{b_2}, \ldots, \bar{b_s})^{-1}$
$= (\bar{0}, \bar{a_2}, \ldots, \bar{a_s}) + (\bar{0}, \overline{-b_2}, \ldots, \overline{-b_s})$
$= (\bar{0}, \overline{a_2 - b_2}, \ldots, \overline{a_s - b_s}) \in H$
$\therefore H \leq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_s}$
Since $1 < |H| = 1 n_2 n_3 \ldots n_s < n_1 n_2 \ldots n_s = |\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_s}|$ we know $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_s}$ is NOT Simple.
$\therefore$ G is not Simple.

So G is Simple iff $G \cong \mathbb{Z}_p$ f.s. prime p $\blacksquare$

B.) *Classify all Abelian groups of size 36 & 540.*

1.) $|G| = 36 = 2^2 \cdot 3^2$

2	3
2, 0	2, 0
1, 1	1, 1

<u>G is isomorphic to one of the following:</u>

a.) $\mathbb{Z}_{2^2 \cdot 3^2} = \mathbb{Z}_{36}$
b.) $\mathbb{Z}_{2^2 \cdot 3} \times \mathbb{Z}_3 = \mathbb{Z}_{12} \times \mathbb{Z}_3$
c.) $\mathbb{Z}_{2 \cdot 3^2} \times \mathbb{Z}_2 = \mathbb{Z}_{18} \times \mathbb{Z}_2$
d.) $\mathbb{Z}_{2 \cdot 3} \times \mathbb{Z}_{2 \cdot 3} = \mathbb{Z}_6 \times \mathbb{Z}_6$

2.) $|G| = 540 = 2^2 \cdot 3^3 \cdot 5$

2	3	5
2, 0, 0	3, 0, 0	1, 0, 0
1, 1, 0	2, 1, 0	
	1, 1, 1	

<u>G is isomorphic to one of the following:</u>

a.) $\mathbb{Z}_{2^2 \cdot 3^3 \cdot 5} = \mathbb{Z}_{540}$
b.) $\mathbb{Z}_{2^2 \cdot 3^2 \cdot 5} \times \mathbb{Z}_3 = \mathbb{Z}_{180} \times \mathbb{Z}_3$
c.) $\mathbb{Z}_{2^2 \cdot 3 \cdot 5} \times \mathbb{Z}_3 \times \mathbb{Z}_3 = \mathbb{Z}_{60} \times \mathbb{Z}_3 \times \mathbb{Z}_3$
d.) $\mathbb{Z}_{2 \cdot 3^3 \cdot 5} \times \mathbb{Z}_2 = \mathbb{Z}_{270} \times \mathbb{Z}_2$
e.) $\mathbb{Z}_{2 \cdot 3^2 \cdot 5} \times \mathbb{Z}_{2 \cdot 3} = \mathbb{Z}_{90} \times \mathbb{Z}_6$
f.) $\mathbb{Z}_{2 \cdot 3 \cdot 5} \times \mathbb{Z}_{2 \cdot 3} \times \mathbb{Z}_3 = \mathbb{Z}_{30} \times \mathbb{Z}_6 \times \mathbb{Z}_3$

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Sec 5.4 #A

A.) Classify all groups of size $5^2 \cdot 7$

LEMMA: If $|G| = 5^2 \cdot 7$, then G is Abelian

Pf: Assume $|G| = 5^2 \cdot 7$

$\Rightarrow \exists P \in Syl_5(G) \wedge \exists Q \in Syl_7(G)$ by Sylow 1

CLAIM 1: $P \cap Q = \{1_G\}$

Pf: Since $P \leq G \wedge Q \leq G$, we know $P \cap Q \leq G$

$\Rightarrow P \cap Q \leq P \wedge P \cap Q \leq Q$

$\Rightarrow |P \cap Q| \mid 25 \wedge |P \cap Q| \mid 7$

$\Rightarrow |P \cap Q| = 1$

$\Rightarrow P \cap Q = \{1_G\}$

CLAIM 2: $P \trianglelefteq G \wedge Q \trianglelefteq G$

Pf: a.) $n_5(G) \equiv 1 \pmod 5 \wedge n_5(G) \mid 7$

$\Rightarrow n_5(G) = 1$, since $7 \not\equiv 1 \pmod 5$

$\Rightarrow P \trianglelefteq G$

b.) $n_7(G) \equiv 1 \pmod 7 \wedge n_7(G) \mid 25$

$\Rightarrow n_7(G) = 1$, since $5, 25 \not\equiv 1 \pmod 7$

$\Rightarrow Q \trianglelefteq G$

CLAIM 3: G is Abelian

Pf: a.) $|G/P| = \frac{175}{25} = 7$

$\Rightarrow G/P \cong \mathbb{Z}_7$ is cyclic (therefore Abelian)

$\Rightarrow G' \leq P$, since $P \trianglelefteq G$

b.) $|G/Q| = \frac{175}{7} = 25 = 5^2$

$\Rightarrow G/Q$ is Abelian

$\Rightarrow G' \leq Q$, since $Q \trianglelefteq G$

By (a) & (b), $G' \leq P \cap Q = \{1_G\}$

$\Rightarrow G' = \{1_G\}$

$\therefore$ G is an Abelian Group $\boxtimes$

By the Lemma, we find all Abelian groups of size $5^2 \cdot 7$

5	7
2,0	1,0
1,1	

G is isomorphic to one of the following:

1.) $\mathbb{Z}_{5^2 \cdot 7} = \mathbb{Z}_{175}$

2.) $\mathbb{Z}_{5 \cdot 7} \times \mathbb{Z}_5 = \mathbb{Z}_{35} \times \mathbb{Z}_5$

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Sec 6.1 #A, B

A.) Let G be a p-Group. Prove that G is Solvable.

Pf: Let p be prime. Let $|G| = p^\alpha$ f.s. $\alpha \geq 1$. Induct on α.

BASIC: $\alpha = 1$ (so $|G| = p$)

$\Rightarrow G \cong \mathbb{Z}_p$ is Cyclic (& therefore Abelian)

$\Rightarrow$ G is Solvable (since any factor group of an Abelian group is Abelian)

INDUCTIVE: Assume the statement is true $\forall\ 1 \leq k < \alpha$. Let $|G| = p^\alpha$.

By Lagrange & the Class Eqn, we know $|Z(G)| \mid p^\alpha$ & $|Z(G)| > 1$

$\Rightarrow |Z(G)| = p^{\alpha - a}$ f.s. $0 \leq a \leq \alpha - 1$

CASE 1: $a = 0$

$\Rightarrow |Z(G)| = p^{\alpha - 0} = p^\alpha = |G|$

$\Rightarrow$ G is Abelian

$\Rightarrow$ G is Solvable

CASE 2: $1 \leq a \leq \alpha - 1$

i.) $Z(G)$ is Abelian

$\Rightarrow Z(G)$ is Solvable

ii.) $Z(G) \trianglelefteq G$

$\Rightarrow G/Z(G)$ is a Group of size $p^{\alpha - (\alpha - a)} = p^a < p^\alpha = |G|$

$\Rightarrow G/Z(G)$ is Solvable by the Inductive Hypothesis

$\therefore$ G is Solvable by (i) & (ii).

$\therefore$ If $|G| = p^\alpha$ for any $\alpha \geq 1$, then G is Solvable ▨

B.) Prove a group of size $3^3 \cdot 11^4$ is Solvable.

Pf: Assume $|G| = 3^3 \cdot 11^4$

$\Rightarrow \begin{cases} \exists P \in Syl_{11}(G) \text{ by Sylow 1} \\ n_{11}(G) \equiv 1 \pmod{11} \wedge n_{11}(G) \mid 27 \text{ by Sylow 3} \end{cases}$

$\Rightarrow n_{11}(G) = 1$ (since $3, 9, 27 \not\equiv 1 \pmod{11}$)

$\Rightarrow P \trianglelefteq G$

a.) P is Solvable (since P is an 11-group)

b.) $|G/P| = \frac{3^3 \cdot 11^4}{11^4} = 3^3$

$\Rightarrow G/P$ is a 3-group

$\Rightarrow G/P$ is Solvable

By (a) & (b), G is Solvable ▨