STAIRCASE TILINGS AND LATTICE PATHS

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Abstract

We define a combinatorial structure, a tiling of the staircase in the $\mathbb{R}^2$ plane, that will allow us, when restricted in different ways, to create direct bijections to Dyck paths of length $2n$, Motzkin paths of lengths n and $n-1$, as well as Schröder paths and little Schröder paths of length n.

Keywords: Tilings, Dyck paths, Motzkin paths, Schröder paths, little Schröder paths.

AMS Mathematical Subject Classifications: 52C20, 05A05, 05A15.

1. Introduction

We define a staircase tiling which can be recognized as a visualization of a combinatorial object enumerated by the Catalan numbers and exhibit direct bijections between staircase tilings and Dyck paths, Motzkin paths, Schröder paths and little Schröder paths. We also enumerate staircase tilings with certain properties using these bijections.

Before introducing our structure, we will give some basic definitions. A *lattice path* of length n is a sequence of points $P_1, P_2, \ldots, P_n$ with $n \geq 1$ such that each point P_i belongs to the plane integer lattice and consecutive points P_i and P_{i+1} are connected by a line segment. We will consider lattice paths in $\mathbb{Z}^2$ whose permitted step types are up-steps $U = (1,1)$, down-steps $D = (1,-1)$, and two types of horizontal steps, $h = (1,0)$ and $H = (2,0)$. We will focus on paths that start from the origin and return to the x-axis, and never pass below the x-axis. An up-step followed by a down-step, UD, is called a *peak*, and a down-step followed by an up-step, DU, is referred to as a *valley*. The part of a lattice path between consecutive visits to the x-axis is called a *block*.

A *Dyck path of length* $2n$ is a lattice path that only uses up-steps and down-steps, and we denote the set of Dyck paths by $\mathcal{D}_n$. Paths composed of up-steps, down-steps and short horizontal steps h are called *Motzkin paths of length* n, and we denote the set of such paths by $\mathcal{M}_n$. Paths of length $2n$ that use up-steps, down-steps and large horizontal steps H are called *Schröder paths of length* $2n$, and the set of Schröder paths is denoted by $\mathcal{S}_n$. It is well-known that Dyck paths of length $2n$ are enumerated by the the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, (see [4, A000108] and [5]), whose ordinary generating function $C(x)$ satisfies the relation

$$xC^2(x) - C(x) - 1 = 0. \tag{1.1}$$

Also, $|\mathcal{M}_n| = M_n$, the n-th Motzkin number (see [4, A000106]), and $|\mathcal{S}_n| = r_n$, the n-th (large) Schröder number (see [4, A006318]). We will also encounter little Schröder paths, which are Schröder paths without peaks at level one (also called *low peaks* or *hills*).

We will now define the staircase structure. Let $\mathcal{A}_n$ be the set in the $\mathbb{R}^2$ plane consisting of all points (x, y) such that $0 \le x \le n$ and $0 \le y \le \lfloor n - x \rfloor + 1$. Figure 1 shows the staircases of size $n = 1, 2, 3$.

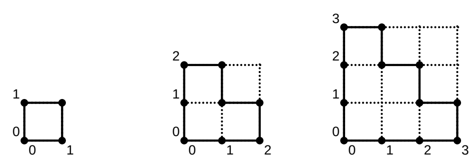

FIGURE 1. The staircases $\mathcal{A}_n$ for $n = 1, 2, 3$.

We refer to the set of points of the staircase $\mathcal{A}_n$ between the lines $y = j - 1$ and $y = j$ as the j-th *row* of $\mathcal{A}_n$ for $1 \le j \le n$ and to the line $x = 0$ of the staircase $\mathcal{A}_n$ as the *border* of $\mathcal{A}_n$. Clearly, $\mathcal{A}_n$ has exactly n rows.

A *row-tiling* of the staircase $\mathcal{A}_n$ is a tiling in which each row of the staircase $\mathcal{A}_n$ is tiled with rectangular tiles of size $1 \times m$, $m \ge 1$, which we will call tiles of *size* or *length* m. Since we are mostly interested in the tiles of size $m \ge 2$, we will refer to those as *large* tiles. We will call a row which has only tiles of size 1 a *row without large tile*, and a row in which a tile fills the row completely (i.e., in row i, the tile is of size $n + 1 - i$) a *complete row*. A row-tiling is said to be *border* if it is a row-tiling such that there is at most one large tile in each row of $\mathcal{A}_k$, and the large tile (if it exists) is adjacent to the staircase border. We will refer to the tile adjacent to the border as the *border tile*. A border row-tiling is said to be *heap* if no large tile is above a smaller tile, i.e., for any two rows i and j with tiles of size p and q, respectively, $i < j$ implies $q < p$. If a row-tiling is both border and heap, then it is said to be a *BHR-tiling* (Border Heap Row-tiling). The set of all border row-tilings of $\mathcal{A}_n$ is denoted by BR_n, and the set of BHR-tilings is denoted by BHR_n. Figure 2 shows the six border row-tilings of the staircase $\mathcal{A}_3$, of which all but the last one are heap.

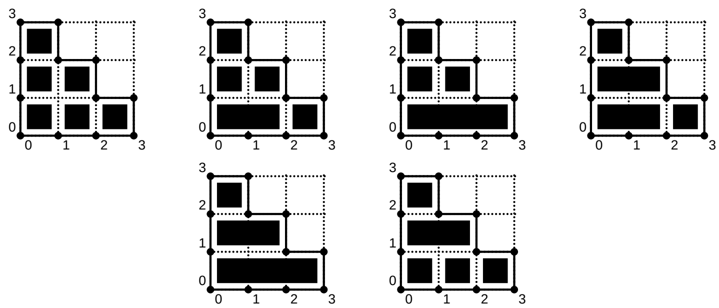

FIGURE 2. Border row-tilings of the staircase $\mathcal{A}_3$.

In Section 2, we will show a direct bijection between BHR_n and $\mathcal{D}n$ and enumerate several statistics on the staircase tilings. In Section 3, we will exhibit bijections between two differently restricted sets of BHR-tilings and $\mathcal{M}_{n-1}$ and $\mathcal{M}_n$, respectively. Finally, we will show a bijection between (large) Schröder paths and colored BHR-tilings in Section 4, and give an alternative characterization of little Schröder paths based on the bijection.

2. Dyck Paths

We will now describe an algorithm to create a *tiling path* associated with a BHR-tiling, from which we will create a Dyck path associated with the tiling. The algorithm will be illustrated in Figure 3 with a BHR-tiling of $\mathcal{A}_3$, which also shows the translation of the tiling path into the corresponding Dyck path.

Path Creation Algorithm. To create a tiling path associated with a BHR-tiling, follow these steps:

- Start at position $(1, n)$.
- If the path is at position (i, j) and the border tile in row j ends at $x = k$, then continue the path to (k, j), and then to $(k, j - 1)$. Note that $k \geq i$, since the tiling is heap, and that the path traces the outline of the border tiles.

- Continue until the path is at position $(i, 0)$, then complete the path to position $(n+1, 0)$.

It is easy to see that each tiling creates a unique tiling path, which always starts at $(1, n)$, ends at $(n+1, 0)$, and never crosses the line $y = n + 1 - x$ (since the staircase $\mathcal{A}_n$ is contained in the half plane $y \leq n + 1 - x$). The path always starts with at least one vertical step (from $(1, n)$ to $(1, n-1)$ since the top row necessarily has a border tile of size 1, and ends with at least one horizontal step (from $(n, 0)$ to $(n+1, 0)$). On the other hand, the tiling path defines the large tiles, and the remainder of the staircase tiling is filled with tiles of size 1, therefore the operation is reversible.

To visually create a Dyck path in $\mathcal{D}_n$ from a tiling path of a staircase tiling of $\mathcal{A}_n$, reflect the tiling path on the line $y = n + 1 - x$, and then rotate counter-clockwise by 45°. Algorithmically, translate each vertical step from (i, j) to $(i, j-1)$ into an up-step U, and each horizontal step from (i, j) to $(i+1, j)$ into a down-step D. Then the tiling path given in Figure 3 translates to the path *UDUUDD*. Since either the visual translation or the algorithmic translation are reversible, this gives a bijection between the tiling paths of BHR-tilings of $\mathcal{A}_n$ and Dyck paths in $\mathcal{D}_n$.

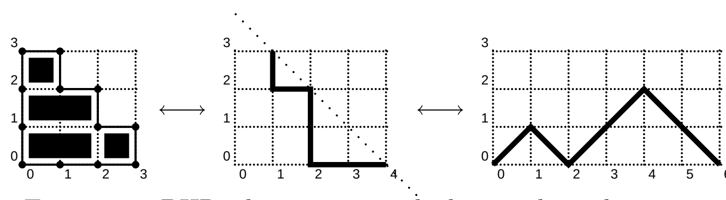

FIGURE 3. BHR-tiling, associated tiling path, and associated Dyck path *UDUUDD* in $\mathcal{D}_n$.

Theorem 2.1. *There is a bijection between BHR_n, the BHR-tilings of $\mathcal{A}_n$, and the set of Dyck paths $\mathcal{D}_n$.*

Note that the BHR-tilings can be interpreted as a visualization of one of the objects enumerated by the Catalan numbers, namely sequences $1 \leq a_1 \leq a_2 \ldots \leq a_n$ such that $a_i \leq i$ (see [6], Exercise 6.19 (s)).

From the Path Creation Algorithm, it is easy to see that every complete row in the BHR-tiling corresponds to a block of the associated Dyck path. A new block of the Dyck path gets created exactly when the tiling path is at position $(n + 1 - j, j)$, necessarily after a horizontal step (except for $j = n$) and before a vertical step. The vertical step indicates that the tile in row j ends at $x = n+1-j$, which means that the tiling has a complete row. Therefore, the number of complete rows of a BHR-tiling equals the number of blocks of the corresponding Dyck path, and we obtain the following corollary.

Corollary 2.2. *The number of BHR-tilings of $\mathcal{A}_n$ with exactly k complete rows is given by $\frac{k}{2n-k}\binom{2n-k}{n}$.*

Proof. Let $C(x)$ denote the generating function of the Catalan numbers (which counts Dyck paths of length $2n$). Then the generating function for the number of Dyck paths of length $2n$ with exactly k blocks is given by $x^kC^k(x)$ from the block decomposition of the Dyck path. Using the Lagrange Inversion Formula (see, for example, [8]) we obtain, after some algebraic simplification, that

$$x^kC^k(x) = \sum_{n\geq 0} \frac{k}{2n+k}\binom{2n+k}{n}x^{n+k} = \sum_{n\geq k} \frac{k}{2n-k}\binom{2n-k}{n}x^n,$$

which gives the desired result. □

We can also enumerate staircase tilings with certain properties using the bijection to Dyck paths. For example, the number of Dyck paths that have their leftmost peak after an even number of steps is enumerated by the Fine numbers (see [4, A006318], [2]). This leads to the following result.

Corollary 2.3. *The number of BHR-tilings of $\mathcal{A}_n$ that start with an even number of border tiles of size 1 are enumerated by the Fine numbers.*

Looking at the changes in the size of the tiles from row to row, we obtain two results related to Narayana numbers, $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ (see [4, A001263]), and Motzkin numbers. Let $c_m(T)$ be the number of *changes of size m* in the tiling T, where a change of size m occurs if the difference in the size of the border tiles in rows i and $i+1$ is m, where $i = 1, \ldots, n-1$. Note that a change of size $m > 1$ creates an occurrence of DU, or a *valley*, in the corresponding Dyck path. Since the number of Dyck paths of length $2n$ with exactly k valleys is counted by the Narayana numbers [3], we obtain the following result.

Corollary 2.4. *The number of tilings in BHR_n with exactly k changes (of any size), i.e., tilings where $\sum_{m=1}^{n-1} c_m(T) = k$, is given by $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$.*

More specifically, whenever a change of size one occurs, the corresponding Dyck path has an occurrence of *UDU*. Callan [1] enumerated Dyck paths according to the number of *UDU*'s, and we therefore obtain a formula for the number of tilings with exactly k changes of size one. For more details on the statistic *UDU* see [7] and references therein.

Corollary 2.5. *The number of tilings in BHR_n with exactly k changes of size one equals $\binom{n-1}{k}M_{n-1-k}$, where M_n denotes the n-th Motzkin number.*

Note that for $k = 0$, we obtain that the number of BHR-tilings without changes of size one are given by M_{n-1}. This connection to the Motzkin numbers has peaked our interest in finding a direct translation from tiling paths to Motzkin paths. Since Motzkin paths enumerate Dyck paths with certain properties, we will translate these properties into properties of the staircase tilings, and then create direct bijections between those staircase tilings and Motzkin paths of length $n - 1$ and n, respectively.

3. Motzkin Paths

We will exhibit two different bijections between border row-tilings of $\mathcal{A}_n$ with certain restrictions and Motzkin paths of length $n - 1$ and n, respectively. In order to do so, we define two operations on staircase tilings, the *reduction* R and the *split* operation. Reduction of a tiling (without complete row of length $m \geq 2$) means taking away the tiles of size one at the end of each row. For example, let T_i be the single tiling of $\mathcal{A}_i$ consisting entirely of tiles of size one, then $R(T_2) = T_1$. The operation split applies to tilings T of $\mathcal{A}_n$ that have a complete row of length $m \geq 2$. If the topmost complete row of length $m \geq 2$ is at row j, $1 \leq j \leq n - 1$, then we cut horizontally above row j, and vertically at $x = n - j$. This creates two smaller tilings T_u (upper tiling) and T_l (lower tiling), of $\mathcal{A}_{n-(j-1)}$ and $\mathcal{A}_{j-1}$, respectively, where the line $x = n - j$ becomes the border of T_l, as illustrated in Figure 4.

We now describe the first bijection based on BHR-tilings without a change of size one.

Motzkin Path Creation Algorithm I. To create a Motzkin path associated with BHR-tilings that have no change of size one, use the following recursive algorithm:

- $\rho(T_1) = \emptyset$.
- If T has no complete row with tile of size $m > 2$, then $\rho(T) = h\rho(R(T))$.
- If T has at least one complete row with tile of size $m > 2$, then $\rho(T) = U\rho(R(T_u))D\rho(T_l)$.
- Apply the algorithm until the tiling has been transformed completely.

Proposition 3.1. *The Motkin Path Creation Algorithm I creates a bijection between BHR-tilings of $\mathcal{A}_n$ that have no change of size one and Motzkin paths of length $n - 1$.*

Proof. Clearly, paths created from different tilings are distinct. We now prove by induction that a tiling T of $\mathcal{A}_n$ creates a Motzkin path of length

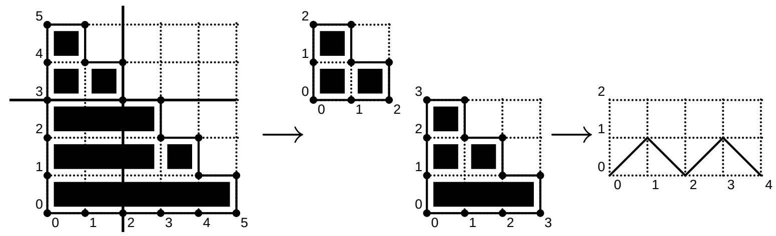

FIGURE 4. Split operation and associated Motzkin path created by Algorithm I.

$n-1$. The steps created by the algorithm are of the correct form for Motzkin paths. We need to show that the path created returns to the x-axis and does not pass below the x-axis. For n=1, the algorithm produces an empty path, which is a Motzkin path of length zero. We now assume the hypothesis for tilings of $\mathcal{A}_k$, $k \leq n$. Let T be a tiling of $\mathcal{A}_{n+1}$. If T has no complete row of size $m > 2$, then the path created consists of a horizontal step followed by the Motzkin path created from $R(T)$. Since $R(T)$ is a tiling of $\mathcal{A}_n$, by hypothesis, $\rho(R(T))$ is a path of length $n-1$, and the overall path is of length n. Prepending the horizontal step does not create a path that passes below the x-axis. If T has a complete row of size $m > 2$, then the Motzkin path is composed of an up-step, a Motzkin path of length $(n-j+2)-1-1$, a down-step, and a Motzkin path of length $j-2$. Overall, the length of the path is n, as desired and, as before, the path does not pass below the x-axis. Thus, the algorithm creates a Motzkin path in $\mathcal{M}_{n-1}$ from a BHR-tiling of $\mathcal{A}_n$.

For the inverse operation, obviously, $\rho^{-1}(\emptyset) = T_1$. For Motzkin paths of length $n \geq 1$, we define the *extension operation* E, which reverses the operation R, i.e., adds a tile of size one to each row. If the Motzkin path is of the form $P_n = hP_{n-1}$, then $\rho^{-1}(P_n) = E(\rho^{-1}(P_{n-1}))$. If the Motzkin path starts with an up-step, then break the path into the first block and the remainder path, i.e., $P_n = UP_kDP_{n-k-2}$. Then $\rho^{-1}(P_n)$ is created in the obvious way from $T_u = E(\rho^{-1}(P_k))$ and $T_l = \rho^{-1}(P_{n-k-2})$.

Note that the condition of not having a change of size one is encoded in the fact that ρ is defined only for tilings having complete rows of length $m > 2$. Any tile that has a change of size one will eventually lead to the tiling $\hat{T}_2$ of $\mathcal{A}_2$ consisting of a tile of size one and one tile of size two, since reduction and splits do not change the relative sizes of the border tiles in the smaller tilings T_u and T_l created along the way. Furthermore, ρ^{-1} never creates $\hat{T}_2$, since this tiling is composed of $T_u = T_l = T_1$ since $T_1 = E(\emptyset)$, which does not get created by ρ^{-1}. $\square$

Note that Corollary 2.4 already proves that a bijection exists, but that Proposition 3.1 exhibits a specific bijection. We now show a second bijection, between Motzkin paths and BHR-tilings that have at most two border tiles of the same size. Note that the algorithm builds the path sequentially (easier to read for humans), but we will use a recursive algorithm in the proof of Proposition 3.2.

Motzkin Path Creation Algorithm II. To create a Motzkin path associated with a BHR-tiling that has at most two border tiles of the same size $m \geq 1$, start reading from the top row of the tiling. Create a Motzkin path that starts at $(0, 0)$ by looking at two cases:

- If rows $i > 2$ and $i - 1$ both contain a border tile of size m, i.e., form a *pair of size* m, and row $i - 2$ contains a border tile of size $m + k$, then append U followed by $k - 1$ Ds to the path. Move to row $i - 2$.
  If $i = 2$, append U followed by $n - m$ Ds to the path and stop.
- If row $i > 1$ contains a border tile of size m, and row $i-1$ contains a border tile of size $m + k$, then append h followed by $k - 1$ Ds to the path. Move to row $i - 1$.
  If $i = 1$, then append h followed by $n - m$ Ds to the path and stop.

Figure 5 shows a BHR-tiling of $\mathcal{A}_5$ and the Motzkin path created by the algorithm. The two top rows (row 5 and 4) contain a border tile of size one. The next row contains a border tile of size two, thus $k = 1$, so the two top rows create an up-step U. For rows 3 and 2, the difference in the respective border tiles is also one, but now the second rule applies and we append a horizontal step h to the path. For the last two rows $m = 3$, and we append *UDD* to the path.

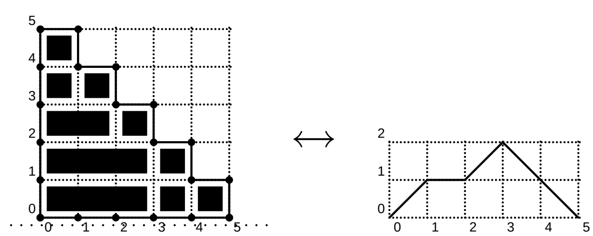

FIGURE 5. Tiling and its associated Motzkin path.

Proposition 3.2. *The BHR-tilings of $\mathcal{A}_n$ that have at most two border tiles of the same size m $(1 \leq m \leq n - 1)$ are in one-to-one correspondence to the Motzkin paths of size n. Moreover, the bijection implies that the* *number of complete rows in the tiling equals the number of blocks in the associated Motzkin path.*

Proof. Clearly, paths created from different tilings are distinct and the only steps created are those for Motzkin paths. In order to prove the statement by induction on n, the size of the triangle $\mathcal{A}_n$, we need a recursive description of the path creation. To do so, we define the *contraction* operation C, where $C(T)$ is created from T by deleting the first column of T and the tiles of size one at the end of each row, as illustrated in Figure 6.

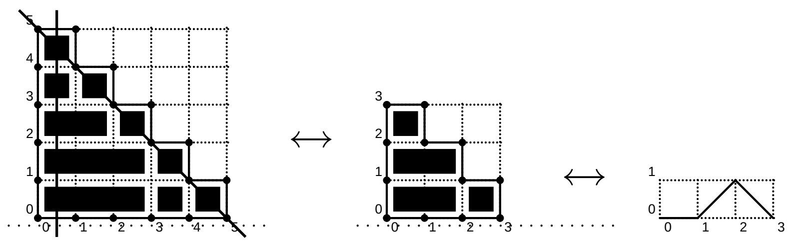

FIGURE 6. Tiling T without complete row of length $m \geq 2$, its tiling $C(T)$ and the Motzkin path for $C(T)$.

Let $\theta(T)$ be the path created sequentially by Algorithm II. Define $\theta(\emptyset) = \emptyset$ and $\theta(T_1) = h$. Based on the sequential algorithm, we distinguish two cases and claim:

(1) If T has no complete row of length ≥ 2, then $\theta(T) = U\theta(C(T))D$;
(2) If T has a complete row of length $m \geq 2$, then $\theta(T) = \theta(T_u)\theta(T_l)$, where T_u and T_l are created by the split operation (see Figure 4).

We will prove both the recursive description of θ and the theorem by induction on n. From Algorithm II, $\theta(T_1) = h$, the single Motzkin path of length 1. The tiling T_1 has exactly one complete row, and the Motzkin path h has exactly one block. Thus the statement is true for $n = 1$. Now assume the statement is true for $k \leq n$ and let T be a tiling of $\mathcal{A}_{n+1}$ with at most two border tiles of size m, $m \geq 1$. In case (1), since T has no complete row of length $m \geq 2$, it necessarily starts with a pair of size 1, followed by a row with a border tile of size 2, thus $\theta(T)$ has to start with an up-step. Furthermore, $\theta(T)$ ends with a down-step since the border tile in the last row has size at most $n - 1$. Thus, $\theta(T)$ is of the form UMD, where M is also a Motzkin path. We need to show that M is created by $\theta(C(T))$. Since M has one less down-step at the end, the tiling from which it is created has a last row that is shorter by one unit. However, this action has to be done across the triangle, hence the deletion of the last tile in each row. M has also one less up-step at the beginning. This up-step results from the configuration of the first two rows, so they need to be deleted, as the sequential algorithm now produces the next step in the path based on the configuration of the rows $n-3$ and $n-4$. Deletion of the first column does not change the relative sizes of the border tiles, and also does not change the structure of the pairs, thus $\theta(C(T))$ produces the proper path, and $\theta(T) = U\theta(C(T))D$ as claimed. (Note that if $n = 2$, then $C(T) = \emptyset$.) Since $C(T)$ is a tiling of $\mathcal{A}_{n-1}$, $\theta(C(T))$ is a Motzkin path of length $n-1$ by induction hypothesis, and $\theta(T) = U\theta(C(T))D$ is a Motzkin path of length $n+1$. The number of complete rows in T is one, and there is exactly one block in the Motzkin path $U\theta(C(T))D$, which shows the second part of the statement. Note that θ can only be reversed uniquely if there are at most two consecutive rows with border tiles of the same length. Otherwise, if there are at least three rows with border tiles of the same length, then eventually a tiling of $\mathcal{A}_3$ consisting of only tiles of size 1 will remain, and θ would produce the same tiling $C(T)$ for both of the tilings in Figure 7.

Figure 7. Uniqueness of θ.

In the second case, when T has a complete row of length $m \geq 2$, we use the split operation and cut above the topmost complete row of length $m \geq 2$. It is clear from the algorithm (since the number of down-steps is based on the difference in size, rather than the actual size of the border tiles in consecutive rows) that $\theta(T) = \theta(T_u)\theta(T_l)$. If the split occurs above row j, then $\theta(T_u)$ and $\theta(T_l)$ produce Motzkin paths of length $(n+1)-j$ and j, respectively. Therefore, $\theta(T) = \theta(T_u)\theta(T_l)$ creates a Motzkin path of length $n+1$, and since the process is reversible, we have shown the bijection. Clearly, # of blocks of $\theta(T)$ = # of blocks of $\theta(T_u)$ + # of blocks of $\theta(T_l)$. On the other hand, the split operation leaves complete rows complete in both T_u and T_l, and thus # of complete rows in T = # of complete rows in T_u + # of complete rows in T_l, which proves the second part of the statement. □

4. Schröder Paths, small and large

We will now add a splash of color to obtain bijections to the Schröder paths and little Schröder paths by introducing colored tiles. Let $CBHR_n$

denote the set of colored BHR-tilings of $\mathcal{A}_n$, where the coloring scheme is as follows:

- Color the tile of size one in the topmost row of the staircase either black or white.
- For any block of k rows where the border tiles are of the same size, color the border tiles in all but the topmost row of the block either black or white.
- All other tiles are colored black.

Figure 8 shows the six colored tilings of $\mathcal{A}_2$.

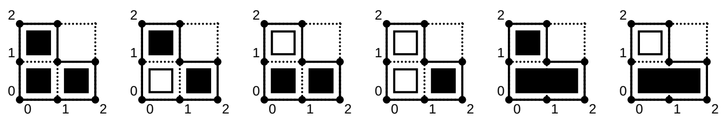

FIGURE 8. The colored tilings of $\mathcal{A}_2$.

Schröeder Path Creation Algorithm. To create a Schröeder path associated with a colored BHR-tiling we use the following recursive algorithm based on the first return decomposition method:

- $\psi(\emptyset) = \emptyset$. If T has no complete row of size $m \geq 2$, let $T_u = T$, $T_l = \emptyset$, else use the split operation (as defined in the algorithm for Motzkin paths) to create the tilings T_u and T_l.
- If the top tile of T_u is colored, then $\psi(T) = \psi(R(T_u))H\psi(T_l)$.
- If the top tile of T_u is not colored, then $\psi(T) = U\psi(R(T_u))D\psi(T_l)$.
- Apply the algorithm until the tiling has been transformed completely.

Figure 9 shows a colored tiling of $\mathcal{A}_5$ and its associated Schröeder path.

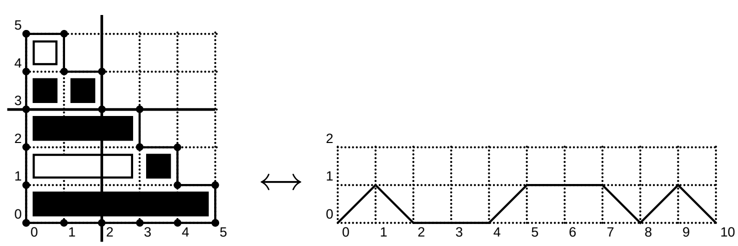

FIGURE 9. A tiling of $\mathcal{A}_5$ and its associated Schröeder path.

Proposition 4.1. *There is a bijection between $CBHR_n$, the set of colored BHR-tilings of $\mathcal{A}_n$, and the set of (large) Schröder paths $\mathcal{S}_n$.*

Proof. The algorithm creates the correct types of steps for a Schröeder path. We need to show that the length of the path is $2n$, that it ends at $(2n, 0)$, and never passes below the x-axis. As for the number of steps, in each application of the algorithm, the size of the staircases is reduced by one (since the reduction operation is applied only to T_u). On the other hand, each application either creates a pair of up- and down-steps or a large horizontal step $(2, 0)$. Thus, a staircase tiling of size n creates a path of length $2n$. We prove the remainder of the conditions, namely # of up-steps = # of down-steps and that the path does not pass below the x-axis, by induction. For $n = 1$, the uncolored tile of size one produces the path UD, and a colored tile of size one produces the path H, both of which satisfy the conditions. Now assume the hypothesis holds for staircases of size $k \leq n$, and consider the path created by a tiling T of $\mathcal{A}_{n+1}$. Note that the algorithm applies ψ to $R(T_u)$ and to T_l, to which the hypothesis applies. If the top tile is not colored, the path is augmented with either a pair of up and down-steps, or with a large horizontal step, which keep the conditions satisfied. Thus the algorithm creates Schröeder paths. The process is reversible, since for a given Schröeder path with a large horizontal step, we use the path on the right of the rightmost horizontal step to create T_l, and the path to the left of the rightmost horizontal step to create T_u. If the Schröeder path has no large horizontal step, then we use the rightmost down-step to uniquely determine T_u and T_l. □

A combinatorial object closely related to the Schröder paths are the little Schröder paths, which are enumerated by the Super-Catalan numbers (see [4, A001003] and [5]). We show a bijection between colored paths with a little less color and the little Schröder paths by no longer coloring the tile in the topmost row. We call tilings colored in this manner *less colored BHR-tilings*. Since this removal of color decreases the number of Schröder paths of length $2n$ by a factor of 1/2 (see for example [5]), we obtain paths that are enumerated by the little Schröder numbers. Since the Schröder paths that are eliminated are exactly those that have no horizontal step H at level zero, we obtain the following corollary and an alternative characterization of little Schröder paths.

Corollary 4.2. *The set of less colored BHR-tilings of $\mathcal{A}_n$ is enumerated by the little Schröder numbers, and is in one-to-one correspondence with Schröder paths without horizontal step H at level zero.*

Emeric Deutsch remarks that the little Schröder numbers count the Schröder paths without peak at level one ([4, A001003]). The equivalence of these paths and those without (large) horizontal step at level zero is immediate, as each peak at level one consists of UD, and can be replaced by H and vice versa.

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