THE MISÈRE ★-OPERATOR

Joint work with Silvia Heubach and Urban Larsson

Matthieu Dufour
University of Quebec at Montreal

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1

The Basics

- A two-player game is called a **combinatorial game** if there is no randomness involved and all possible moves are known to each player.
- A combinatorial game is called **impartial** if both players have the same moves, and **partizan** otherwise.
- Under **normal play**, the last player to move wins. Under **misère play**, the last player to move loses.
- **Examples:**

2

Impartial Games

- Only two possible outcome classes:
  - **P-positions** (**P**revious player wins = losing position)
  - **N-positions** (**N**ext player wins = winning position)

- Characterization of N- and P-positions
  - From a P-position, **all** allowed moves lead to an N-position
  - From an N-position, there is **at least one** move to a P-position.
- In misère play, the **terminal** positions are **N-positions**

3

Subtraction Games

- A **subtraction** or **take-away game** is played on one or more stacks of tokens
- A **subtraction set M** indicates the allowed moves, as long as they do not result in negative stack height(s)
- **Positions** are described as vectors of stack heights

Example:

- **NIM** on one stack **M = {1, 2, 3, …}**
- **WYTHOFF** is played on two stacks. At each turn we can either take one or more tokens from **one** stack, or the **same number from both stacks**.

  M = {(1,0), (2,0),… , (0,1), (0,2),… , (1,1),(2,2),…}

4

Misère ★-Operator

- **Observation**:
  - For subtraction games, positions and allowed moves have the same structure
  - One can iteratively create a new game whose moves are determined by certain positions of the original game.
- The **misère-play ★-operator** is defined as follows:
  - We start with a subtraction game **M** that is described by the allowed moves.
  - We compute the set of P-positions, **P(M)**
  - Then ★ : **M** → **M*** = **P(M)**, that is, the losing positions of **M** become the moves for the game **M***
- Notation: $\mathbf{M}^0 = \mathbf{M}$, $\mathbf{M}^n = (\mathbf{M}^{n-1})^*$
- **M** is **reflexive** if **M** = **M***

5

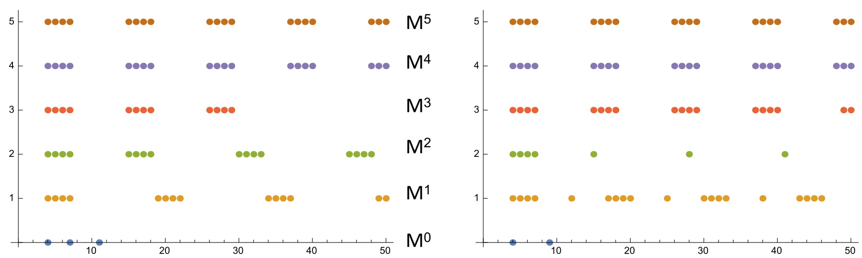
Questions for Misère ★-Operator

- Does the misère-play ★-operator converge (point-wise)?
- Limit games are (by definition) reflexive. What is the structure of reflexive games and/or limit games (if they exist)?

6

Example in one dimension (M ⊂ N)

Misère-play ★-operator applied five times to initial game



$M^0 = \{4, 7, 11\}$ $G^0 = \{4, 9\}$

7

Observations from Example

- Looks like there is convergence (fixed point) for each of the games
- Limit games seem to have a periodic structure: blocks of moves alternate with blocks of non-moves
- $M^0 = \{4, 7, 11\}$ and $G^0 = \{4, 9\}$ seem to have the same limit

Question: What have the two sets M^0 and G^0 in common?

Answer: The minimal element, **k = 4**.

8

Convergence Result

Theorem

Starting from any game $M \subseteq \mathbb{N}^d \setminus \{\mathbf{0}\}$, the sequence of games created by the misère-play ★-operator converges to a (reflexive) limit game M^{∞}

Proof idea: (in dimension d)

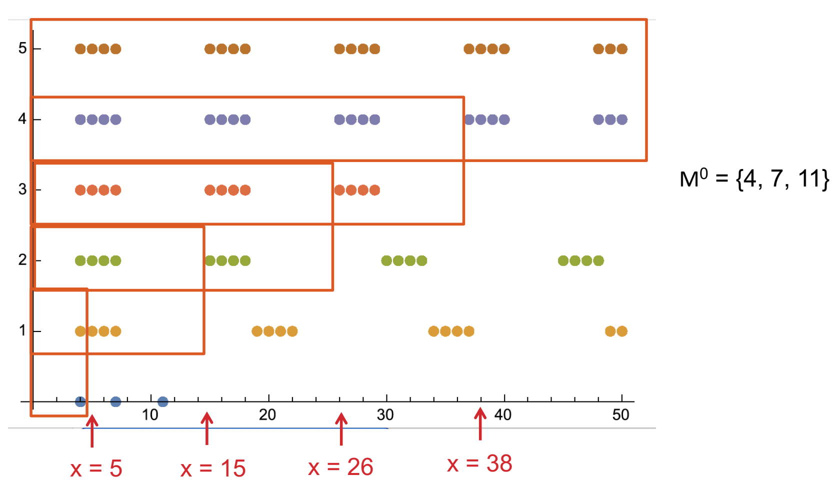
- Positions become fixed either as moves or non-moves from "smaller to larger". There are four possibilities:

	$\in P(M^i) = M^{i+1}$	$\notin P(M^i) = M^{i+1}$
$\in M^i$	Fixed as a move	Erased as move
$\notin M^i$	Introduced as move	Fixed as non-move

- Show that smallest element not yet fixed becomes fixed.

9

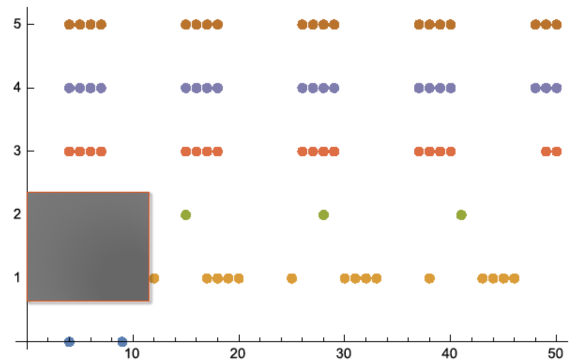
Point-wise convergence



10

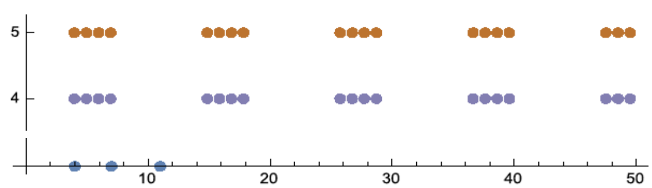
Point-wise convergence

- Not all positions switch from non-move to move:



11

Structure of reflexive games in **one** dimension



$\mathbf{M_k}$:= { i p_k + k, …, i p_k + (2k – 1)| i = 0,1,…}, where p_k = 3k-1

Theorem

The game M $\subseteq \mathbb{N}$ is reflexive if and only if M = M_k for some k > 0.

12

What feature of M determines M^{∞}?

Theorem

Two games $M, G \subseteq \mathbb{N}^d \setminus \{\mathbf{0}\}$ have the same limit game if and only if their unique **sets of minimal elements** (with the usual partial order on $\mathbb{N}^d$) are the same.

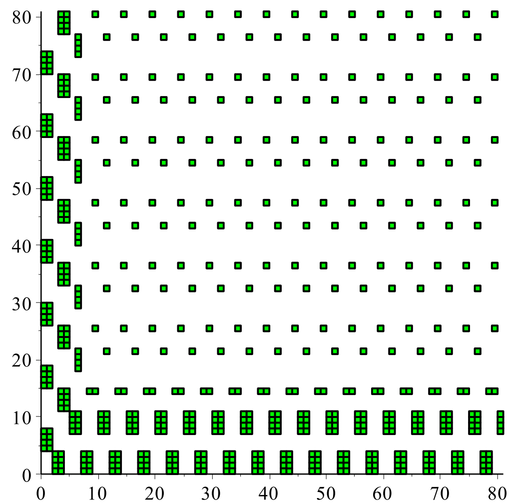
13

Structure of limit games in **two** dimensions

- Much less is known, but we think similar structure results hold in general
- The limit games seem to have **periodic** structure
- In two dimensions, positions/moves are points in the two-dimensional integer lattice

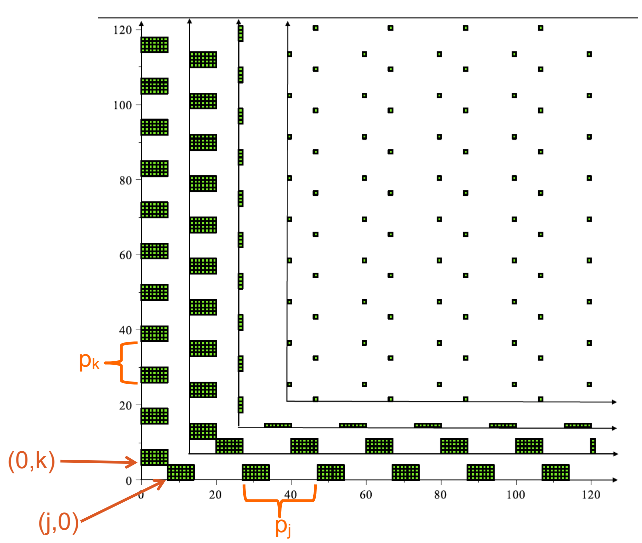
14

Example in two dimensions



15

Example of game $M_{j,k}$



16

Reflexivity of $M_{j,k}$

Theorem

The game $\mathbf{M}_{j,k} \subsetneq \mathbb{N}^2 \setminus \{\mathbf{0}\}$ is reflexive.

Corollary

The limit game of a set $M \subsetneq \mathbb{N}^d \setminus \{\mathbf{0}\}$ equals the game $\mathbf{M}_{j,k}$ if and only if the set of minimal elements of **M** equals {(j,0),(0,k)}.

17

References for Normal Play

- U. Larsson, P. Hegarty, and A. S. Fraenkel. Invariant and dual subtraction games resolving the Duchêne-Rigo conjecture, *Theoretical Computer Science*, 412, pp 729-735, 2011.

- U. Larsson. The *-operator and invariant subtraction games. *Theoretical Computer Science*, 422, pp 52-58, 2012.

18

THANK YOU!

sheubac@calstatela.edu

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19