Star-extremal Circulant Graphs

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Abstract

A graph is called star-extremal if its fractional chromatic number is equal to its circular chromatic number (also known as the star chromatic number). We prove that members of a certain family of circulant graphs are star-extremal. The result generalizes some known theorems of Sidorenko [18] and Gao and Zhu [10]. Then we show relations between circulant graphs and distance graphs and discuss their star-extremality. Furthermore, we give counter-examples to two conjectures of Collins [6] on asymptotic independence ratios of circulant graphs.

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1 Introduction

Given a positive integer n and a set $S \subseteq \{1, 2, 3, \cdots, \lfloor n/2 \rfloor\}$, let $G(n, S)$ denote the graph with vertex set $V(G) = \{0, 1, 2, \cdots, n-1\}$ and edge set $E(G) = \{uv : |u-v|_n \in S\}$, where $|x|_n := \min\{|x|, n - |x|\}$ is the *circular distance modulo* n. Then $G(n, S)$ is called the *circulant graph* of order n with the *generating set* S.

Circulant graphs have been investigated in different fields. Such graphs are "star-polygons" to geometers [7]. The well-known Ádám's conjecture [1] states: $G(n, S)$ and $G(n, S')$ are isomorphic if and only if $S' = kS = \{ks : s \in S\}$ for some unity k in the ring Z_n. Alspach and Parsons [2] proved that this conjecture does not hold in general. However, it is true for some special classes of circulant graphs. Parsons [15] characterized the set A_k of connected circulant graphs $G(n, S)$ such that the neighbors $N(x)$ for each vertex x induce a k-cycle in $G(n, S)$. Then Ádám's conjecture was established for circulant graphs in A_k.

In this article, we explore the star-extremality of circulant graphs. A graph is called *star-extremal* if its fractional chromatic number and circular chromatic number, defined below, are equal.

A *fractional coloring* of a graph G is a mapping c from $\mathcal{I}(G)$, the set of all independent sets of G, to the interval $[0, 1]$ of real numbers such that $\sum \{c(I) : x \in I \text{ and } I \in \mathcal{I}(G)\} \geq 1$ for any vertex x in G. The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the *weight*, $w(c) = \sum \{c(I) : I \in \mathcal{I}(G)\}$, of a fractional coloring c of G. For a different but equivalent definition of the fractional chromatic number, we refer the reader to [17].

Let k and d be positive integers such that $k \geq 2d$. A (k, d)-coloring of a graph $G = (V, E)$ is a mapping c from V to $\{0, 1, \cdots, k-1\}$ such that $|c(x) - c(y)|_k \geq d$ for any edge xy in G. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of k/d for which there exists a (k,d)-coloring of G. The circular chromatic number is also known as the *star-chromatic number* in the literature [20].

For any graph G, it is well-known [22] that

$$\max\{\omega(G), \tfrac{|V(G)|}{\alpha(G)}\} \le \chi_f(G) \le \chi_c(G) \le \chi(G) \quad \text{and} \quad \lceil \chi_c(G) \rceil = \chi(G), \qquad (*)$$

where $\omega(G)$ is the *clique number* (i.e., the maximum number of vertices of a complete subgraph in G); $\alpha(G)$ is the *independence number* (i.e., the maximum number of vertices of an independent set in G.) Hence, a graph G is star-extremal if the equality holds in the second inequality in (*).

The notion of star-extremality for graphs arose from the study of the chromatic number and the circular chromatic number of the lexicographic product of graphs. The lexicographic product $G[H]$ of G and H is the graph with vertex set $V(G)\times V(H)$ and in which $(v_1,w_1)(v_2,w_2)$ is an edge if and only if $v_1v_2 \in E(G)$ or $v_1 = v_2$ and $w_1w_2 \in E(H)$ (Informally, we substitute a copy of H for each vertex of G.) It was proved in [10] that, if G is star-extremal, then $\chi_c(G[H]) = \chi_c(G)\chi(H)$ for any graph H. Therefore for any star-extremal graph G, the circular chromatic number, and hence the chromatic number, of the lexicographic product $G[H]$ is determined by $\chi_c(G)$ and $\chi(H)$. Klavžar [12] also used star-extremal graphs to investigate the chromatic numbers of lexicographic products of graphs.

The star-extremality for circulant graphs was first discussed by Gao and Zhu [10]. They proved that if all the vertices of $G(n,S)$ have degree ≤ 3, then $G(n,S)$ is star-extremal. On the other hand, there exist non star-extremal circulant graphs (see Section 3 below or [10].) In general, it seems a difficult problem to determine whether or not an arbitrary circulant graph is star-extremal.

As circulant graphs are vertex-transitive, we know that

$$\chi_f(n,S) = n/\alpha(n,S), \qquad (**)$$

where $\chi_f(n, S)$ and $\alpha(n, S)$ denote, respectively, the fractional chromatic number and the independence number for $G(n, S)$. Therefore the determination of the independence number of a circulant graph is equivalent to the determination of its fractional chromatic number. Recently, Codenotti et al. [5] have proved that it is NP-complete to compute the independence number for general circulant graphs. However, for some circulant graphs including the ones discussed in this article, the independence number can be computed in polynomial time.

In Section 2, we focus on the family of circulant graphs whose generating set S consists of consecutive integers. Given integers $k < k' \leq n/2$, let $S_{k,k'}$ denote the set $\{k, (k+1), \cdots, k'\}$. We will determine the exact value of $\alpha(n, S_{k,k'})$ for any $n \geq 2k'$ and $k' \geq (5/4)k$. This result is used to prove that the circulant graphs $G(n, S_{k,k'})$ are star-extremal for all $n \geq 2k'$ and $k' \geq (5/4)k$.

For some special values of k, k', and n, the circulant graphs $G(n, S_{k,k'})$ have appeared in several articles. It was proved in [10] that $G(n, S_{1,k'})$ (i.e., $k = 1$) is star-extremal for all $n \geq 2k'$. When $k' = 2k - 1$ and $n \geq 4k - 1$, the circulant graph $G(n, S_{k,2k-1})$ is a triangle-free regular graph with degree $2k$. Sidorenko [18] proved that $\alpha(n, S_{k,2k-1}) = 2k$ for $6k - 2 \leq n \leq 8k - 3$ and applied this result to answer a question of Erdős [16], namely, the existence of triangle-free regular graphs on $n(\neq 3, 7, 9)$ vertices with its independence number equal to the degree. Gao and Zhu [10] then applied Sidorenko's result to show that the circulant graphs $G(n, S_{k,2k-1})$ are star-extremal for $6k - 2 \leq n \leq 8k - 3$.

Let Z denote the set of all integers. For a given finite set S of positive integers, the *distance graph*, denoted by $G(Z, S)$, has Z as its vertex set and uv forms an edge if $|u - v| \in S$. Thus the distance graph $G(Z, S)$ can be viewed as the limit of circulant graphs $G(n, S)$ as n approaches infinity. In Section 3, we explore the relation of star-extremality between circulant graphs and distance graphs for general sets S.

The *independence ratio* of a graph G is defined to be $\alpha(G)/|V(G)|$. In Section 4, we show that for a given S, the fractional chromatic number of the distance graph $G(Z,S)$ is equal to the reciprocal of the asymptotic independence ratio of circulant graphs $G(n,S)$ as n approaches infinity. Applying this fact, we present counter-examples to two conjectures of Collins [6] on the asymptotic independence ratio of circulant graphs.

2 Circulant graphs with interval generating sets

We shall discuss the star-extremality of circulant graphs whose generating sets are of the form $S_{k,k'} = \{k, (k+1), \cdots, k'\}$, where $k < k' \leq n/2$. For the case that $k' \geq (5/4)k$, we determine the exact values of $\alpha(n, S_{k,k'})$ for all n. Using this result, we show that such circulant graphs are star-extremal.

One of the tools we shall use is the following *multiplier method*, which was first used in [10] and has been applied to solve problems concerning coloring of circulant graphs as well as distance graphs [3, 19]. Given a circulant graph $G(n,S)$ and a positive integer t, let

$$\lambda_t(n,S) := \min\{|ti|_n : i \in S\},$$

and let

$$\lambda(n,S) := \max\{\lambda_t(n,S) : t = 1, 2, 3, \cdots\},$$

where the multiplications ti are carried out modulo n and $|x|_n$ is the circular distance modulo n. For any positive integer t, the mapping c on $\{0, 1, 2, \cdots, n-1\}$ defined by $c(i) = ti$ is an $(n, \lambda_t(n,S))$-coloring for $G(n,S)$ (multiplications are carried out modulo n.) Hence, $\chi_c(n,S) \leq n/\lambda(n,S)$. Combining this with (*) and (**), we obtain the following result.

Lemma 1 *([10]) Let $G(n, S)$ be a circulant graph. Then $\lambda(n, S) \leq \alpha(n, S)$. Moreover, if $\lambda(n, S) = \alpha(n, S)$, then $\chi_f(n, S) = \chi_c(n, S) = n/\alpha(n, S)$, i.e., $G(n, S)$ is star-extremal.*

The value of $\lambda(n, S)$ can be calculated in polynomial time. To be precise, we have the following:

Lemma 2 *Let $G(n, S)$ be a circulant graph. Then $\lambda(n, S) = \lambda_t(n, S) = |ts|_n = |t(-s)|_n$ for some t, $1 \leq t \leq \lceil n/2 \rceil$, and $s \in S$.*

Proof. By definition, $\lambda_t(n, S) = \lambda_{n-t}(n, S) = \lambda_{t'}(n, S)$ for any $t \equiv t' \pmod{n}$, and $|ts|_n = |t(-s)|_n$. Q.E.D.

Sidorenko [18] proved that $\alpha(n, S_{k,2k-1}) = 2k$ if $6k - 2 \leq n \leq 8k - 3$. Later on, Gao and Zhu [10] proved that $\lambda(n, S_{k,2k-1}) = 2k$ under the same condition on n. Combining these two results with Lemma 1, the following was obtained in [10].

Theorem 3 *If $k' = 2k - 1$, then the circulant graphs $G(n, S_{k,k'})$ are star-extremal for all n, where $6k - 2 \leq n \leq 8k - 3$.*

Other special sub-families of the circulant graphs $G(n, S_{k,k'})$ that have been studied include the following two.

Theorem 4 *([10]) If $k' \leq n/2$, then $G(n, S_{1,k'})$ is star-extremal and $\chi_f(n, S_{1,k'}) = \chi_c(n, S_{1,k'}) = n/\lfloor \frac{n}{k'+1} \rfloor$.*

Theorem 5 *([10]) Suppose $k' = k + l \leq n/2$. If $n - 2k' < \min\{k, l\}$, then $G(n, S_{k,k'})$ is star-extremal and $\chi_f(n, S_{k,k'}) = \chi_c(n, S_{k,k'}) = n/k$.*

The proofs of Theorems 4 and 5 are obtained, respectively, by showing $\alpha(n, S_{1,k'}) = \lambda(n, S_{1,k'}) = \lfloor \frac{n}{k'+1} \rfloor$ and $\alpha(n, S_{k,k'}) = \lambda(n, S_{k,k'}) = k$ under the assumptions on k and k'.

We note here that the circulant graphs $G(n, S_{1,k'})$ in Theorem 4 are indeed powers of the cycle C_n on n vertices. Given n and r, the r-th *power* of C_n, denoted by C_n^r, has the same vertex set as C_n, and u, v are adjacent if their distance on the cycle C_n is not greater than r. Therefore $G(n, S_{1,k'}) = C_n^{k'}$ by definition.

In their study of the circular chromatic number of planar graphs Gao, Wang, and Zhou [9] defined a family of planar graphs Q_n, called ***triangular prisms***, which have vertex set $V = \{u_0, u_1, u_2, \cdots, u_{n-1}\} \cup \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, and edge set E consisting of two n-cycles $(u_0, u_1, \cdots, u_{n-1})$ and $(v_0, v_1, \cdots, v_{n-1})$ and $2n$ edges (u_i, v_i), (u_{i+1}, v_i) for every $0 \leq i \leq n-1$ $(u_0 = u_n)$. In [9], the argument to compute the values of $\chi_f(Q_n)$ is long. The proof can be shortened considerably by applying known results in circulant graphs. The family of planar graphs Q_n are precisely the second powers of even cycles. Indeed, $(v_0, u_1, v_1, u_2, v_2, \cdots, u_{n-1}, v_{n-1}, u_0)$ is a cycle of length $2n$, and $Q_n \cong C_{2n}^2$. Hence, by Theorem 4, $\chi_f(Q_n) = \chi_c(Q_n) = 2n/\lfloor \frac{2n}{k+1} \rfloor$.

Now we consider the general family of circulant graphs $G(n, S_{k,k'})$. We view the vertices of $G(n, S_{k,k'})$ as circularly ordered in the clockwise direction, and denote by $[a, b]$ the set of integers $\{a, a+1, a+2, \cdots, b\}$, where the addition is taken under modulo n. For example, $[2, 5] = \{2, 3, 4, 5\}$ and $[5, 2] = \{5, 6, \cdots, n-1, 0, 1, 2\}$.

Lemma 6 *Suppose I is an independent set of $G(n, S_{k,k'})$. Then, for any j, the cardinality of $I \cap [j, j+k+k'-1]$ is at most k.*

Proof. By symmetry, it suffices to show that for any independent set I, the cardinality of $I \cap [0, k+k'-1]$ is at most k. Suppose $i \in [0, k+k'-1]$ is the least element of I. Then $i+k, i+k+1, \cdots, i+k' \notin I$. Let $A = [i+1, i+k-1] \cap I$ and $B = [i+k'+1, k'+k-1] \cap I$. If $x \in A$, then $x + k' \notin B$. This implies $|A| + |B| \leq k-1$. Therefore $|I \cap [0, k+k'-1]| \leq k$. Q.E.D.

Lemma 7 *Suppose* $G = G(n, S_{k,k'})$ *with* $n = q(k + k') + r$, $0 \le r \le k + k' - 1$. *Then*

$$\lambda(G) \ge \begin{cases} \lambda_q(G) = qk & \text{if } 0 \le r \le k'; \\ \lambda_{q+1}(G) = qk + r - k' & \text{if } k' + 1 \le r \le k' + k - 1. \end{cases}$$

Proof. It suffices to show that $\lambda_q(G) = qk$ when $0 \le r \le k'$ and $\lambda_{q+1}(G) = qk+r-k'$ when $k' + 1 \le r \le k' + k - 1$.

If $0 \le r \le k'$, then

$$\begin{aligned} \lambda_q(G) = \min\{ \quad & qk, q(k+1), q(k+2), \cdots, qk', \\ & n - qk, n - q(k+1), n - q(k+2), \cdots, n - qk' \}. \end{aligned}$$

Because $qk \le q(k+1) \le \cdots \le qk'$ and $n - qk \ge n - q(k+1) \ge n - q(k+2) \ge \cdots \ge n - qk'$, it is enough to show $n - qk' \ge qk$. This is true since $n - qk' = qk + r \ge qk$.

If $k' + 1 \le r \le k' + k - 1$, then

$$\begin{aligned} \lambda_{q+1}(G) = \min\{ \quad & (q+1)k, (q+1)(k+1), (q+1)(k+2), \cdots, (q+1)k', \\ & n - (q+1)k, n - (q+1)(k+1), \cdots, n - (q+1)k' \}. \end{aligned}$$

Because $(q+1)k \le (q+1)(k+1) \le (q+1)(k+2) \le \cdots \le (q+1)k'$ and $n-(q+1)k \ge n-(q+1)(k+1) \ge n-(q+1)(k+2) \ge \cdots \ge n-(q+1)k' = qk+r-k'$, it is enough to show $qk+r-k' \le (q+1)k-1$. This is true since $qk+r-k' \le qk+(k+k'-1)-k' = (q+1)k - 1$. Q.E.D.

Theorem 8 *Suppose* $G = G(n, S_{k,k'})$ *and* $k' \ge (5/4)k$. *Let* $n = q(k + k') + r$, *where* $0 \le r \le k + k' - 1$. *Then*

$$\alpha(G) = \lambda(G) = \begin{cases} qk & \text{if } 0 \le r \le k'; \\ qk + r - k' & \text{if } k' + 1 \le r \le k' + k - 1. \end{cases}$$

Equivalently, $\alpha(G) = \lambda(G) = qk + \max\{0, r - k'\}$.

Proof. Let $n = q(k + k') + r$, $0 \le r \le k + k' - 1$. By Lemmas 1 and 7, it suffices to show that $\alpha(G) \le qk + \max\{0, r - k'\}$. If $q = 0$, the result follows from Lemma 6. Thus we may assume $q \ge 1$.

Assume to the contrary that $\alpha(G) > qk + \max\{0, r - k'\}$. Let I be a maximum independent set of G. Regard I as a disjoint union of I-intervals, where an *I-interval* is a maximal interval $[a, b]$ consisting of vertices in I. Then the length (namely, the number of vertices) of any I-interval is between 1 and k. By Lemma 6 and the assumption that $q \geq 1$, there are at least two I-intervals. Assume that the independent set I chosen has the minimum number of I-intervals among all maximum independent sets of G.

Two I-intervals $[a, b]$ and $[c, d]$ are called *consecutive* if $[b + 1, c - 1] \cap I = \emptyset$. Note that the consecutive "relation" is not symmetric, i.e., $[a, b]$ and $[c, d]$ being consecutive does not imply that $[c, d]$ and $[a, b]$ are consecutive. (Indeed, $[c, d]$ and $[a, b]$ are not consecutive if $[a, b]$ and $[c, d]$ are consecutive and I contains more than two I-intervals.) For two consecutive I-intervals $[a, b]$ and $[c, d]$, the cardinality of the set $[b + 1, c - 1]$ is called the ***gap*** between them.

First we show that if $[a, b]$ and $[c, d]$ are two consecutive I-intervals, then $b + k' + 1 \leq c + k - 1$ (or equivalently, $b - k + 1 \leq c - k' - 1$). Here we assume, without loss of generality, that $0 \leq a \leq b < c \leq d \leq n - 1$. Suppose to the contrary that $b + k' + 1 \geq c + k$. For any $b + 1 \leq x \leq c - 1$, if y is adjacent to x then straightforward calculations show that y is adjacent either to b or to c. Hence none of the neighbors of x is in I. This implies that the set $I' = I \cup [b + 1, c - 1]$ is independent with $|I'| > |I|$, which contradicts our choice of I.

Next we show that the gap between any two consecutive I-intervals is at most $k - 2$. Suppose to the contrary that there exist consecutive I-intervals $[a, b]$ and $[c, d]$ such that $|[b + 1, c - 1]| \geq k - 1$. Since $[a, b] \subseteq I$, it follows from the definition of $S_{k,k'}$ that $[a + k, b + k'] \cap I = \emptyset$. Hence $[b + 1, b + k'] \cap I = \emptyset$. We partition the interval $[b + k' + 1, b]$ $(= [0, n - 1] - [b + 1, b + k'])$ into sub-intervals of length $k + k'$, except the last sub-interval which may have size less than $k + k'$ (when $r \geq k' + 1$.) If $0 \leq r \leq k'$, then the number of such sub-intervals is equal to q. By Lemma 6, $|I| \leq qk$, which is contrary to our assumption. If $k' + 1 \leq r \leq k + k' - 1$, then the number of such sub-intervals is equal to $q + 1$, and the last interval has size $r - k'$. Again, it follows from Lemma 6 that $|I| \leq qk + r - k'$, which is contrary to our assumption. Therefore, the gap between any two consecutive I-intervals is at most $k-2$.

Now we show that the gap between any two consecutive I-intervals is greater than $2(k'-k)$. Assume to the contrary that $[a,b]$ and $[c,d]$ are consecutive I-intervals with gap t, $t \leq 2(k'-k)$. Let

$$I' = (I \cup [b+1, c-1]) - ([b+k'+1, c+k-1] \cup [b-k+1, c-k'-1]).$$

It is clear that I' is an independent set of G with $|I'| \geq |I| + t - 2(t-(k'-k)) \geq |I|$. Hence, I' is a maximum independent set with less intervals than I, which is contrary to our assumption.

We conclude that the gap between any two consecutive I-intervals is between $2(k'-k)+1$ and $k-2$. In particular, this implies that $2(k'-k)+1 \leq k-2$, for otherwise, we have already arrived at a contradiction. Now by the assumption that $k' \geq (5/4)k$, the gap between any two consecutive I-intervals is between $\frac{k}{2}+1$ and $k-2$. This implies that any set of k consecutive vertices in G intersects exactly two I-intervals.

For any consecutive I-intervals $[a,b]$ and $[c,d]$, we claim that $|[a,b]| + |[c,d]| \leq \frac{k}{2} - 1$. First we note that $|[a,d]| \leq k$. For otherwise, we would have $d \geq a+k$. This implies $c \geq b+k'+1$ since $[a+k, b+k'] \cap I = \emptyset$. Then the gap between $[a,b]$ and $[c,d]$ would be greater than $k-2$, a contradiction. It follows that $|[a,b]| + |[c,d]| \leq k - 2(k'-k) - 1 \leq \frac{k}{2} - 1$ since $k' \geq (5/4)k$.

Now choose two consecutive I-intervals $[a,b]$ and $[c,d]$ such that $|[a,b]| + |[c,d]|$ is the largest among all pairs of consecutive I-intervals. Let $[u,v]$ be the I-interval preceding $[a,b]$ (i.e., $[u,v]$ and $[a,b]$ are consecutive I-intervals) and let $[x,y]$ be the I-interval following $[c,d]$. Since $|I| > qk \geq k$ and since the union of any two consecutive I-intervals contains at most $\frac{k}{2} - 1$ vertices, we know that there are at least five I-intervals. So the intervals $[a,b]$, $[c,d]$, $[u,v]$, and $[x,y]$ are distinct.

We now show that $[x,y]$ (respectively, $[u,v]$) is the only I-interval included in

$[b+k'+1, c+k-1]$ (respectively, in $[b-k+1, c-k'-1]$). Because $[a,b], [c,d] \subseteq I$, we have $([a+k, b+k'] \cup [c+k, d+k']) \cap I = \emptyset$. In addition, by the arguments above, $[a,b]$, $[c,d]$ and $[c,d]$, $[x,y]$ are the only two I-intervals included in $[a, a+k-1]$ and $[c, c+k-1]$, respectively. Hence $[x,y] \subseteq [b+k'+1, c+k-1]$ and $[x,y]$ is the only I-interval included in $[b+k'+1, c+k-1]$. Similarly, we can show that $[u,v]$ is the only I-interval included in $[b-k+1, c-k'-1]$.

According to the choice of $[a,b]$ and $[c,d]$, we have $|[u,v]| \leq |[c,d]|$ and $|[x,y]| \leq |[a,b]|$. Therefore $|[u,v]| + |[x,y]| \leq |[a,b]| + |[c,d]| \leq \frac{k}{2} - 1$. Let

$$I' = (I \cup [b+1, c-1]) - ([u,v] \cup [x,y]).$$

By the discussion in the previous paragraph, it is clear that I' is an independent set with

$$|I'| \geq |I| + \frac{k}{2} + 1 - (\frac{k}{2} - 1) > |I|,$$

which contradicts our maximality assumption about I. Q.E.D.

Corollary 9 *If $k' \geq (5/4)k$, then $G(n, S_{k,k'})$ is star-extremal.*

Theorem 10 *Suppose $G = G(n, S_{k,k'})$ with $n = q(k+k') + r$, $0 \leq r \leq k+k'-1$, and*

$$q > \frac{1}{k-k'} - \frac{k'}{k+k'} - \frac{kk'}{(k-k')(k+k')}.$$

Then G is star-extremal. Moreover, the values of $\alpha(G)$ and $\lambda(G)$ are the same as in Theorem 8.

Proof. Let I be a maximum independent set of G. The I-intervals are similarly defined as in the proof of Theorem 8. Assume that the chosen set I has the minimum number of I-intervals among all maximum independent sets of G. Then the gap between any two consecutive I-intervals, as shown in the proof of Theorem 8, is between $2(k'-k)+1$ and $k-2$.

In the following, we show that for any i, $|I \cap [i, i+k+k'-1]| \le 2k - k'$. Let a be the least element of $I \cap [i, i+k+k'-1]$ and let $[a, b]$ be the first non-empty intersection of an I-interval with $[i, i+k+k'-1]$. Note that when $a = i$, $[a, b]$ may be a part of an I-interval. Similarly, let $[c, d]$ be the last non-empty intersection of an I-interval with $[a, a+k]$. By the proof of Theorem 8, $[a, b]$ and $[c, d]$ are the two I-intervals included in $[a, a+k-1]$, so $[a, b] \ne [c, d]$.

In addition, it is true that $I \cap [c-(k'-k), c-1] = \emptyset$ since the gap between any two consecutive I-intervals is at least $2(k'-k) > k'-k$. Because $I \cap ([c+k, d+k'] \cup [a+k, a+k']) = \emptyset$, the following is clear.

$$I \cap [i, i+k+k'-1] \subset I \cap [a, a+k+k'-1] = [c, d] \cup (I \cap (A \cup B)),$$

where $A = [a, c-(k'-k)-1] \cup [d+1, a+k-1]$ and $B = [a+k', c+k-1] \cup [d+k'+1, a+k+k'-1]$. For each vertex $x \in A$, we have $x + k' \in B$. So if $x \in I \cap A$, then $x + k' \in B - I$, and vice versa. This one-to-one correspondence implies that $|I \cap (A \cup B)| \le |A| = |B|$. Therefore, $|I \cap [i, i+k+k'-1]| \le |A| + |[c, d]| = k - (k'-k) = 2k - k'$.

For each $0 \le i \le n-1$, let $n_i = |I \cap [i, i+k+k'-1]|$. Then

$$(k+k')|I| = \sum_{i=0}^{n-1} n_i \le n(2k-k').$$

Now if $0 \le r \le k'$, then by Lemmas 1 and 7, it suffices to show that $\alpha(G) = |I| \le qk$. Assume to the contrary that $|I| \ge qk+1$. Since $n \le q(k+k') + k'$, we have

$$(k+k')(qk+1) \le (q(k+k')+k')(2k-k').$$

A contradiction emerges after simplifying this inequality.

If $k' < r \le k+k'-1$. It sufffices to show that $|I| \le qk + r - k'$. If $|I| \ge qk + r - k' + 1$, then

$$(k+k')(qk+r-k'+1) \le (q(k+k')+r)(2k-k').$$

This inequality leads to a contradiction, too. Q.E.D.

3 Circulant graphs and distance graphs

Circulant graphs and distance graphs are closely related. Given a finite set S of positive integers, the distance graph $G(Z,S)$ can be viewed as the limit of the sequence of circulant graphs $G(n,S)$ as n approaches infinity. Therefore, for a given S, if $G(n,S)$ is star-extremal for all n, then $G(Z,S)$ is star-extremal. However, the reverse of this implication is not always true. Take $S = \{1,3,4,5\}$. It is known [3] that $G(Z,S)$ is star-extremal, while it was proved [10] that $\chi_f(10,S) = 5 < \chi_c(10,S) = 6$. So $G(10,S)$ is not star-extremal.

In this section, we prove that, for a given S, if the distance graph $G(Z,S)$ is star-extremal, then there exist infinitely many n such that the circulant graphs $G(n,S)$ are star-extremal.

The fractional chromatic number $\chi_f(Z,S)$ for a distance graph $G(Z,S)$ has very close connections with T-coloring [13] and an earlier number theory problem about the density of sequences with missing differences. For references about these connections, we refer the reader to [4, 11, 14]. Among the results in [4], it is proved that $\chi_f(Z,S)$ always exists and is a rational number for any finite S.

A *homomorphism* (or edge-preserving map) from a graph G to another graph H is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ if $uv \in E(G)$. If such a homomorphism exists, we say that G admits a homomorphism to H and denote this by $G \to H$. If $G \to H$, then we have $\chi_f(G) \leq \chi_f(H)$ and $\chi_c(G) \leq \chi_c(H)$ by composition of functions.

Lemma 11 *For a given S, $G(Z,S) \to G(n,S)$ for all $n \geq 2\max S$, where $\max S$ denotes the largest member of the set S.*

Proof. Define a mapping $f : Z \to [0, n-1]$ by $f(x) = x \bmod n$. It is easy to verify that f is a homomorphism from $G(Z,S)$ to $G(n,S)$. Q.E.D.

Corollary 12 *For a given* S, $\chi_c(Z,S) \le \chi_c(n,S)$ *and* $\chi_f(Z,S) \le \chi_f(n,S)$ *for all* $n \ge 2\max S$.

Theorem 13 *If* $G(Z,S)$ *is star-extremal for a given* S*, then there exists a positive integer* m *such that* $G(km,S)$ *is star-extremal for any positive integer* k.

Proof. By a result in [4], we may assume $\chi_c(Z,S) = \chi_f(Z,S) = p/q$ for some rational number p/q. Let $d = \max S$. According to Corollary 12, it is enough to show that, for some $m \ge 2d$, there exists a (p,q)-coloring for any $G(km,S)$ because we would then have

$$p/q = \chi_f(Z,S) \le \chi_f(mk,S) \le \chi_c(mk,S) \le p/q.$$

Since $\chi_c(Z,S) = p/q$, by a result in [10], there exists a (p,q)-coloring $f : Z \to [0, p-1]$ of $G(Z,S)$. Partition non-negative integers into blocks such that each block consists of p^d consecutive vertices. Consider the restriction of f to these blocks. By the pigeonhole principle, there exist two blocks with the same color sequence. Let x and y be the leading vertices of these two blocks such that $x < y$. Then $f(x+i) = f(y+i)$ for $0 \le i \le p^d - 1$. Let $m = y - x$. Define the mapping $f'(j) = f(x+j)$ for $0 \le j \le m-1$. It is clear that f' is a (p,q)-coloring for $G(m,S)$.

For $k \ge 2$, define a mapping $f'' : [0, km-1] \to [0, p-1]$ by $f''(v) = f'(v \bmod m)$. It is clear that f'' is a (p,q)-coloring for $G(km,S)$. Q.E.D.

4 Independence ratio

In this section, we discuss relations between the independence ratio and the fractional chromatic number of circulant graphs and distance graphs. Based on these relations, we give counter-examples to two conjectures of Collins [6].

Let $S = \{a_1, a_2, \cdots, a_l\}$ be a set of positive integers with $a_1 < a_2 < \cdots < a_l$. In her study of the asymptotic independence ratio of the circulant graphs $G(n, S)$, Collins introduced the S-*graph*, denoted by $G(S)$, which has vertex set $V = \{0, 1, 2, \cdots, a_1 + a_l - 1\}$ and edge set $E = \{uv : |u - v| \in S\}$. Note that $G(S)$ is not necessarily a circulant graph.

Given n and S, let $\mu(n, S) := \alpha(n, S)/n$ and $\mu(S) := \alpha(G(S))/(a_1 + a_l)$ denote the independence ratio of the circulant graph $G(n, S)$ and the S-graph $G(S)$, respectively. The *asymptotic independence ratio* $L(S)$ of a given set S is defined in [6] by

$$L(S) := \lim_{n\to\infty} \mu(n, S).$$

According to (**), we have $\mu(n, S) = 1/\chi_f(n, S)$. Combining this with the fact that $\chi_f(Z, S) = \lim_{n\to\infty} \chi_f(n, S)$, the following result is obtained.

Theorem 14 $L(S) = 1/\chi_f(Z, S)$ *for any given* S.

A set $S = \{a_1, a_2, \cdots, a_l\}$, $a_1 < a_2 < \cdots < a_l$, $l \geq 2$, is called *reversible* if $a_1 + a_l = a_2 + a_{l-1} = \cdots = a_{\lfloor \frac{l}{2} \rfloor} + a_{\lceil \frac{l}{2} \rceil}$. Collins [6] proved that $L(S) = \mu(S)$ if S is reversible and proposed the following:

Conjecture 1 *([6]) Suppose* $S = \{a_1, a_2, \cdots, a_l\}$, $a_1 < a_2 < \cdots < a_l$, $l \geq 2$, *is a reversible set. Then* $\alpha(n, S) = \lfloor n\mu(S) \rfloor$ *for any integer* n *satisfing* $n > a_1 + 2a_l$.

We now give a counter-example to Conjecture 1. The interval set $S_{k,k'}$ studied in Section 2 is reversible. However, by Lemma 6 and Theorem 8, we have $\mu(S_{k,k'}) = k/(k + k')$ and $\alpha(n, S_{k,k'}) \neq \lfloor n\mu(S_{k,k'}) \rfloor$ when $k' \geq (5/4)k$, $n = q(k + k') + r$, and $r \geq k' + 1$.

For a non-reversible set S, Collins [6] gave two methods for constructing reversible sets from S. Let $S = \{a_1, a_2, \cdots, a_l\}$ and let $x = a_{l-1} + a_l$ and $y = a_1 + a_l$. Define $\hat{S} = S \cup (x - S)$ (here $x - S$ is the set $\{x - i \mid i \in S\}$) and $\tilde{S} = S \cup (y - S)$. Collins [6] showed that $L(S) \geq \max\{\mu(\hat{S}), \mu(\tilde{S})\}$ and proposed the following:

Conjecture 2 *([6])* $L(S) = \max\{\mu(\hat{S}), \mu(\tilde{S})\}$.

For a counter-example to this conjecture, take $S = \{1, 2, 3, 6\}$. It is known [13] and easy to see that $\omega(Z, S) = \chi(Z, S) = 4$, so $\chi_f(Z, S) = \chi_c(Z, S) = 4$. Hence $L(S) = 1/4$. But $\hat{S} = \{1, 2, 3, 6, 7, 8\}$, $\tilde{S} = \{1, 2, 3, 4, 5, 6\}$, $\mu(\hat{S}) = 2/9$, and $\mu(\tilde{S}) = 1/7$.

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