①

(a) Consider

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = \vec{0}$$

This is equivalent to

$$c_1(-1+x+x^2) + c_2(x^2) + c_3(-1+x) = 0+0x+0x^2$$

This is equivalent to

$$(-c_1-c_3) + (c_1+c_3)x + (c_1+c_2)x^2 = 0+0x+0x^2$$

This is equivalent to

$$\boxed{\begin{aligned} -c_1 \quad\quad\quad - c_3 &= 0 \\ c_1 \quad\quad\quad + c_3 &= 0 \\ c_1 + c_2 \quad\quad\quad &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_2 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \underbrace{0 \quad 0 \quad 0}_{\text{reduced form}} & & & 0 \end{array}\right)$$

This becomes

$$\begin{aligned} c_1 \qquad + c_3 &= 0 \\ c_2 - c_3 &= 0 \\ 0 &= 0 \end{aligned}$$

leading variables: c_1, c_2
free variable: c_3

$\rightarrow$

Solution:

$c_3 = t$
$c_2 = c_3 = t$
$c_1 = -c_3 = -t$
for any t

Ex: $t = 1$, gives
$c_1 = -1$, $c_2 = 1$, $c_3 = 1$

So, $-v_1 + v_2 + v_3 = \vec{0}$
Thus, v_1, v_2, v_3 are linearly dependent.

(b) Since v_1, v_2, v_3 are linearly dependent
they cannot be a basis.

(2) We want to try and solve

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 & 2c_1 \\ 0 & 2c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 & 0 \\ 0 & c_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 & 2c_1 \\ 0 & 2c_1 + c_2 \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} c_1 + 2c_2 &= 1 \quad (1) \\ 2c_1 &= 2 \quad (2) \\ 0 &= 0 \quad (3) \\ 2c_1 + c_2 &= 1 \quad (4) \end{aligned}}$$

$\longrightarrow$ (2) gives $c_1 = 1$

Plug this into (1) and (4) gives:

(1) $1 + 2c_2 = 1 \rightarrow c_2 = 0$

(4) $2 + c_2 = 1 \rightarrow c_2 = -1$

You cant have $c_2 = 0$ and $c_2 = -1$

Thus there are no solutions and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is <u>not</u> in the span of v_1, v_2

Note: You can also do row reduction and that would work also. See next page

Another way to solve this:

$$\boxed{\begin{aligned} c_1 + 2c_2 &= 1 & (1)\\ 2c_1 \quad &= 2 & (2)\\ 0 &= 0 & (3)\\ 2c_1 + c_2 &= 1 & (4)\end{aligned}}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{array}\right) \xrightarrow[-2R_1+R_4 \to R_4]{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & -1 \end{array}\right)$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[-\frac{1}{3}R_3 \to R_3]{-\frac{1}{4}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \boxed{\begin{aligned} c_1 + 2c_2 &= 1 \\ c_2 &= 0 \\ 0 &= 1/3 \\ 0 &= 0 \end{aligned}}$$

row echelon form

$\boxed{0 = 1/3 \text{ shows no solutions}}$

So, $v \notin \operatorname{span}(\{v_1, v_2\})$

3

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in W$.

Then, $a - 2b + 3c = 0$.

So, $a = 2b - 3c$.

Thus, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2b-3c \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2b \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} -3c \\ 0 \\ c \end{pmatrix}$

$= b\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

So, $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \text{span}\left(\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}\right)$

Note that $2 - 2(1) + 3(0) = 0$ and $-3 - 2(0) + 3(1) = 0$

thus $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \in W$.

So, $W = \text{span}\left(\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}\right)$

Let's show $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent.

Consider $c_1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Then, $\begin{pmatrix} 2c_1 - 3c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Thus, $\boxed{\begin{aligned} 2c_1 - 3c_2 &= 0 \\ c_1 &= 0 \\ c_2 &= 0 \end{aligned}}$

The only solutions are $c_1 = 0$, $c_2 = 0$.

Thus, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent also.

Thus, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ are a basis for W.

Thus, $\dim(W) = 2$.

A / B

A This is HW 1 #2(c)

B This is HW 2 #5

(c) (Method 1) Suppose

$$c_1 w_1 + c_2 w_2 + c_3 w_3 = \vec{0}$$

This is equivalent to

$$c_1(v_1 - v_2) + c_2 v_1 + c_3(v_1 + 3v_2) = \vec{0}$$

This is equivalent to

$$(c_1 + c_2 + c_3)v_1 + (-c_1 + 3c_3)v_2 = \vec{0}$$

Since v_1, v_2 are lin. ind. this forces

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 \qquad + 3c_3 &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 0 & 3 & 0 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right)$$

reduced

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + 4c_3 &= 0 \end{aligned}} \to \boxed{\begin{aligned} c_1 &= -c_2 - c_3 \quad (1) \\ c_2 &= -4c_3 \quad (2) \\ c_3 &= t \quad (3) \end{aligned}} \to \boxed{\begin{aligned} c_3 &= t \\ c_2 &= -4t \\ c_1 &= -4t - t \\ &= -5t \\ t &\in \mathbb{R} \end{aligned}}$$

leading: c_1, c_2
free: c_3

t can be any element of $\mathbb{R}$

<u>Ex:</u> $t = 1 \to c_1 = -5,\ c_2 = -4,\ c_3 = 1$

So, $-5w_1 - 4w_2 + w_3 = \vec{0}$

So, w_1, w_2, w_3 are lin. dep.

(c) (Method 2)

Let $W = \text{span}(\{v_1, v_2\})$.

Since v_1, v_2 are linearly independent we know $\dim(W) = 2$.

Since $w_1 = v_1 - v_2$, $w_2 = v_1$, $w_3 = v_1 + 3v_2$ we know $w_1, w_2, w_3 \in W$.

So, we have 3 vectors in a 2-dimensional vector space.

Since $3 > 2$ by a theorem in class, w_1, w_2, w_3 are linearly dependent.

D (Method 1) ← See next page for method 2

Assume $v_0 \in W$. We can show that W is a subspace via the 3 conditions method.

(i) Since $v_0 \in W$ and W is a subspace, we know $-v_0 \in W$. Thus, we can set $w = -v_0$ and get $w + v_0 = -v_0 + v_0 = \vec{0}$ is in W.

(ii) Let $v_1, v_2 \in W$. Then $v_1 = w_1 + v_0$ and $v_2 = w_2 + v_0$, where $w_1, w_2 \in W$. Then,

$$v_1 + v_2 = w_1 + w_2 + v_0 + v_0$$

Since $w_1, w_2, v_0 \in W$, and W is a subspace, W is closed under $+$ and so, $v_1 + v_2 = w_1 + w_2 + v_0 + v_0$ is in W.

(iii) Let $v \in W$ and $\alpha \in F$. Then $v = w + v_0$ where $w \in W$. Then, $\alpha v = \alpha w + \alpha v_0$.

Since $w, v_0 \in W$ and W is a subspace, we know αw and αv_0 are in W. And again since αw and αv_0 are in W and W is a subspace we know $\alpha w + \alpha v_0 \in W$. So, $\alpha v \in W$.

By (i), (ii), (iii), W is a subspace of V.

(D) (Method 2)

Assume $v_0 \in W$.

One can actually show that $W = W_0$ and then since W is a subspace, so is W_0

$\boxed{W \subseteq W_0}$:

Let $w \in W$.

Then, $w = (w - v_0) + v_0$. Since $v_0 \in W$ and W is a subspace, $-v_0 \in W$.

Since w and $-v_0$ are in W and W is a subspace $w - v_0 \in W$.

Thus, $w = \underbrace{(w - v_0)}_{\text{in } W} + v_0 \in W_0$

So, $W \subseteq W_0$.

$\boxed{W_0 \subseteq W}$

Let $v \in W_0$. Then $v = w + v_0$ where $w \in W$.

Since $w \in W$ and $v_0 \in W$ and W is a subspace we know $w + v_0 \in W$. Thus, $v \in W$.

So, $W_0 \subseteq W$.

Since $W \subseteq W_0$ and $W_0 \subseteq W$ from above we have $W = W_0$ and so W_0 is a subspace of V.