1

Let $x, y \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$.

Then, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

So,

$$T(\alpha x + \beta y) = T\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2 \\ \alpha x_2 + \beta y_2 + \alpha x_3 + \beta y_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 - \alpha x_2 \\ \alpha x_2 + \alpha x_3 \end{pmatrix} + \begin{pmatrix} \beta y_1 - \beta y_2 \\ \beta y_2 + \beta y_3 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_1 - x_2 \\ x_2 + x_3 \end{pmatrix} + \beta \begin{pmatrix} y_1 - y_2 \\ y_2 + y_3 \end{pmatrix}$$

$$= \alpha T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta T\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

2 (a) $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in N(T)$ iff $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\begin{pmatrix} a+2b \\ b-3c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\boxed{\begin{array}{l} a+2b = 0 \\ b-3c=0 \end{array}}$

a & b are leading variables
c is free variable

iff $\boxed{\begin{array}{l} a=-2b \\ b=3c \\ c=t \end{array}}$

iff $\boxed{\begin{array}{l} c=t \\ b=3t \\ a=-2(3t)=-6t \end{array}}$

iff $\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -6t \\ 3t \\ t \end{pmatrix} = t\begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}}$

basis for $N(T)$ is $\beta = \left\{ \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix} \right\}$

(b) $\text{nullity}(T) = \dim(N(T)) = 1$

(c) $\text{rank}(T) = \dim(\mathbb{R}^3) - \dim(N(T))$
$= 3 - 1 = 2$

(d) Yes. $R(T)$ is a 2-dimensional subspace of $\mathbb{R}^2$ which is also 2-dimensional
Thus, $R(T) = \mathbb{R}^2$.

③ (a)

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1+1\\-2+4\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} = 2\cdot\begin{pmatrix}1\\1\end{pmatrix} + 0\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1+2\\-2+8\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix} = 0\cdot\begin{pmatrix}1\\1\end{pmatrix} + 3\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

$$[T]_\beta = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}$$

(b) $\boxed{\text{Method 1}}$

$$[T(z)]_\beta = [T]_\beta [z]_\beta = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}4\\9\end{pmatrix}$$

$\boxed{\text{Method 2}}$

Since $[z]_\beta = \begin{pmatrix}2\\3\end{pmatrix}$ this means $z = 2\cdot\begin{pmatrix}1\\1\end{pmatrix} + 3\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\8\end{pmatrix}$

Thus, $T(z) = T\begin{pmatrix}5\\8\end{pmatrix} = \begin{pmatrix}5+8\\-10+32\end{pmatrix} = \begin{pmatrix}13\\22\end{pmatrix}$

To get $[T(z)]_\beta$ need to solve $\begin{pmatrix}13\\22\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix}$

$$\boxed{\begin{matrix}a+b=13\\a+2b=22\end{matrix}} \xrightarrow{-R_1+R_2\to R_2} \boxed{\begin{matrix}a+b=13\\b=9\end{matrix}} \to \boxed{\begin{matrix}b=9\\a=13-9=4\end{matrix}}$$

So, $[T(z)]_\beta = \begin{pmatrix}4\\9\end{pmatrix}$

(c) By part (b) since $[z]_\beta = (2,3)$ we know that

$$z = 2\cdot\begin{pmatrix}1\\1\end{pmatrix} + 3\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\8\end{pmatrix}$$

And,

$$\begin{pmatrix}5\\8\end{pmatrix} = 5\begin{pmatrix}1\\0\end{pmatrix} + 8\begin{pmatrix}0\\1\end{pmatrix}$$

Thus, $[z]_\gamma = \begin{pmatrix}5\\8\end{pmatrix}$

A or B

A This is HW 3 problem 6(a).

B This is HW 4 problem 4.

(c) We are given that $N(T) = \text{span}(\{1+x+x^2, x+x^2\})$

Let $\beta = \{1+x+x^2, x+x^2\}$

β is a linearly independent set since the only sol. to

$$\underbrace{c_1(1+x+x^2)+c_2(x+x^2)}_{c_1 \cdot 1+(c_1+c_2)\cdot x+(c_1+c_2)x^2} = 0+0x+0x^2+0x^3+0x^4+0x^5+0x^6$$

is $c_1 = 0$, $c_2 = 0$.

So, $\dim(N(T)) = (\text{size of } \beta) = 2$.

Thus,

$\dim(P_6(\mathbb{R})) = \dim(N(T)) + \dim(R(T))$

So, $7 = 2 + \dim(R(T))$.

Thus, $\dim(R(T)) = 5$.

Since $R(T)$ is a subspace of $P_4(\mathbb{R})$ and $\dim(R(T)) = 5 = \dim(P_4(\mathbb{R}))$ we know $R(T) = P_4(\mathbb{R})$ and T is onto.