Topic 0 -
Series

Sequences (HW 0)

Def: A sequence $(z_n)_{n=1}^{\infty}$ is an ordered list of complex number.

Ex: $z_n = 1 + \frac{i}{n}$, $n \geq 1$

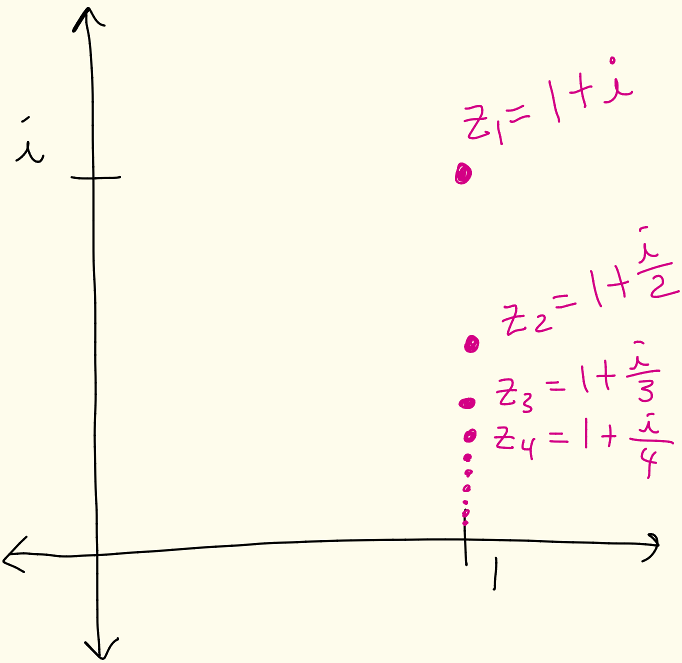
$z_1 = 1 + i$

$z_2 = 1 + \frac{i}{2}$

$z_3 = 1 + \frac{i}{3}$

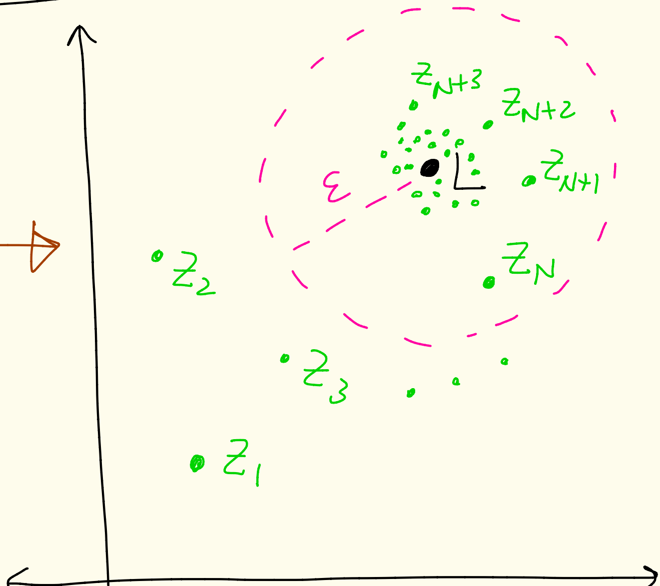
$z_4 = 1 + \frac{i}{4}$

$\vdots$



Def: A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers <u>converges</u> to $L \in \mathbb{C}$ if for every $\varepsilon > 0$ there exists $N > 0$ where if $n \geq N$ then $|z_n - L| < \varepsilon$. If $(z_n)_{n=1}^{\infty}$ converges to L then we write $\lim_{n \to \infty} z_n = L$

N depends on what ε is



(picture for limit def)

Thm: Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers.
Let $L \in \mathbb{C}$.
Suppose $z_n = x_n + i y_n$ for $n \geq 1$.
Suppose $L = X + iY$.
Then $\lim_{n\to\infty} z_n = L$ iff

$$\lim_{n\to\infty} x_n = X \text{ and } \lim_{n\to\infty} y_n = Y$$

Proof in 4680 notes on 10/28/20

Ex: $z_n = 1 + \frac{i}{n}$

$$= 1 + i \cdot \frac{1}{n} = x_n + i y_n$$

where $x_n = 1$, $y_n = \frac{1}{n}$.

From 4650 or Calculus,

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Thus, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i\left(\lim_{n\to\infty} y_n\right) = 1 + i \cdot 0 = 1$

Def: A sequence of complex numbers $(z_n)_{n=1}^{\infty}$ is Cauchy if for every $\varepsilon > 0$ there exists $N > 0$ where if $n, m \geq N$ then $|z_n - z_m| < \varepsilon$

Theorem: A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers is Cauchy iff there exists $L \in \mathbb{C}$ where $\lim_{n\to\infty} z_n = L$.

[Ie Cauchy iff converges]

proof: $(\Leftarrow)$ Suppose $\lim_{n\to\infty} z_n = L$ for some $L \in \mathbb{C}$.

Let $\varepsilon > 0$.

Since $\lim_{n \to \infty} z_n = L$ there exists $N > 0$ where if $n \geq N$ then $|z_n - L| < \frac{\varepsilon}{2}$.

Then if $n, m \geq N$ then

$$
\begin{aligned}
|z_n - z_m| &= |z_n - L + L - z_m| \\
&\leq |z_n - L| + |L - z_m| \\
&= |z_n - L| + |z_m - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

$\boxed{|a-b| = |b-a|}$

So, $(z_n)_{n=1}^{\infty}$ is Cauchy.

($\Rightarrow$) Suppose that $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Let $z_n = x_n + i y_n$

By HW 0 #3, since $(z_n)_{n=1}^{\infty}$ is Cauchy, the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy in $\mathbb{R}$.

From Math 4650 (Analysis I) since $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are Cauchy in $\mathbb{R}$ and $\mathbb{R}$ is complete, there exist $x \in \mathbb{R}$ and $y \in \mathbb{R}$ where $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Let $L = x + iy$. By thm on page 4, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$

$= x + iy = L$ $\square$

Thm: Let $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ be sequences of complex numbers.

1. Suppose $\lim_{n\to\infty} z_n = A$ and $\lim_{n\to\infty} w_n = B$.

   (a) If $\alpha, \beta \in \mathbb{C}$, then
   $$\lim_{n\to\infty} (\alpha z_n + \beta w_n) = \alpha A + \beta B$$

   (b) $$\lim_{n\to\infty} z_n w_n = AB$$

   (c) If $w_n \neq 0$ for all $n \geq 1$ and $B \neq 0$, then $\lim_{n\to\infty} \dfrac{z_n}{w_n} = \dfrac{A}{B}$

2. If $(z_n)_{n=1}^{\infty}$ converges, then the sequence is bounded, that is, there exists $M > 0$ where $|z_n| \leq M$ for all $n \geq 1$.