Topic 2 –

Spanning, linear independence, bases, dimension

Def: Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n \in V$.

1. The **span** of $v_1, v_2, \ldots, v_n$ is defined to be

$$\mathrm{span}(\{v_1, v_2, \ldots, v_n\}) = \Big\{ \underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n}_{\text{is called a linear combination of } v_1, v_2, \ldots, v_n} \;\Big|\; \alpha_1, \ldots, \alpha_n \in F \Big\}$$

2. If $V = \mathrm{span}(\{v_1, v_2, \ldots, v_n\})$ then we say that $v_1, v_2, \ldots, v_n$ **span** V or we say that $v_1, v_2, \ldots, v_n$ is a **spanning set** for V.

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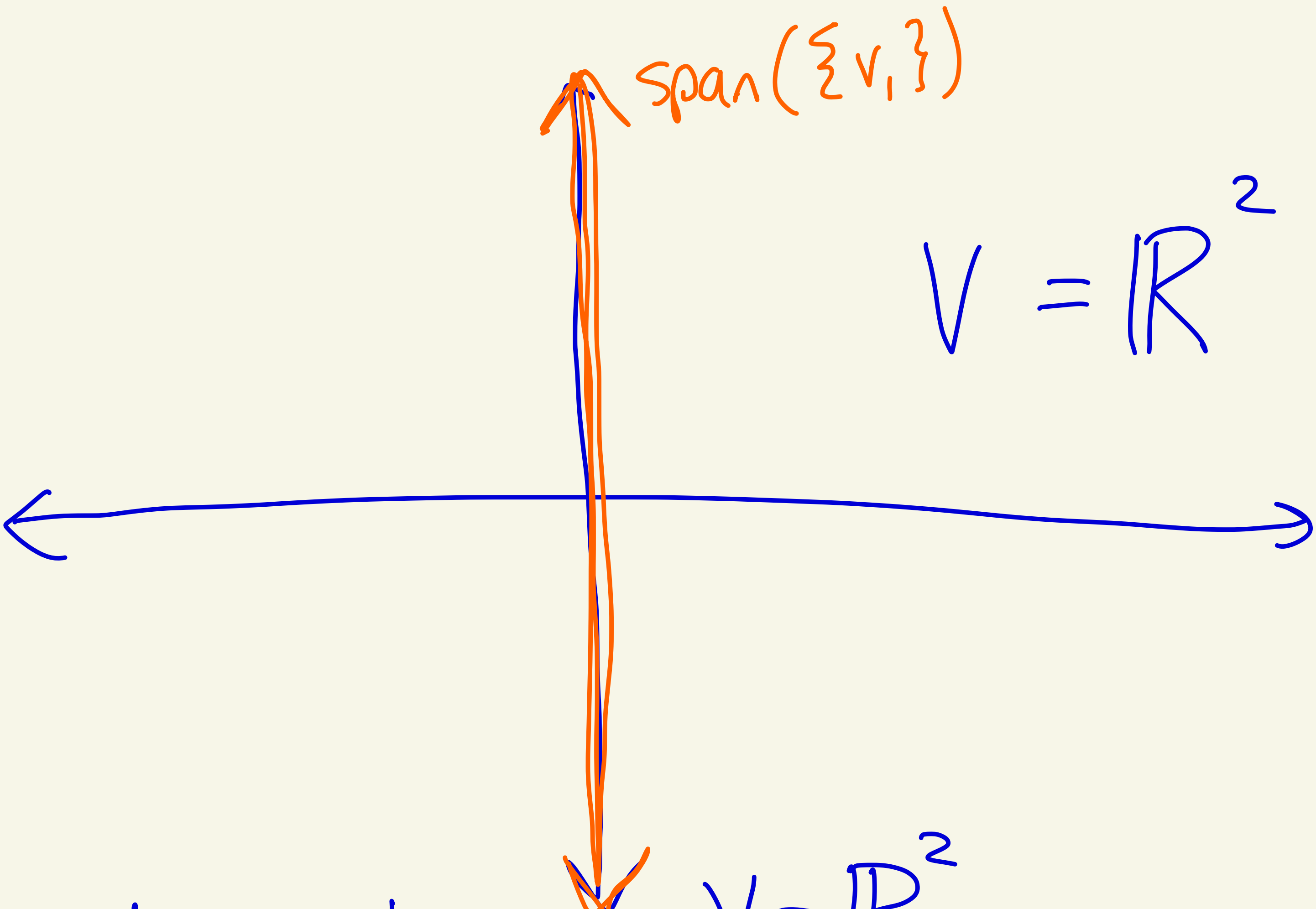
Ex: $V=\mathbb{R}^2$, $F=\mathbb{R}$

Let $v_1=(0,1)$.

Then,

$$\begin{aligned}\operatorname{span}(\{v_1\}) &= \{\alpha_1 v_1 \mid \alpha_1\in\mathbb{R}\}\\ &= \{\alpha_1(0,1) \mid \alpha_1\in\mathbb{R}\}\\ &= \{(0,\alpha_1) \mid \alpha_1\in\mathbb{R}\}\end{aligned}$$



v_1 does not span $V=\mathbb{R}^2$.

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

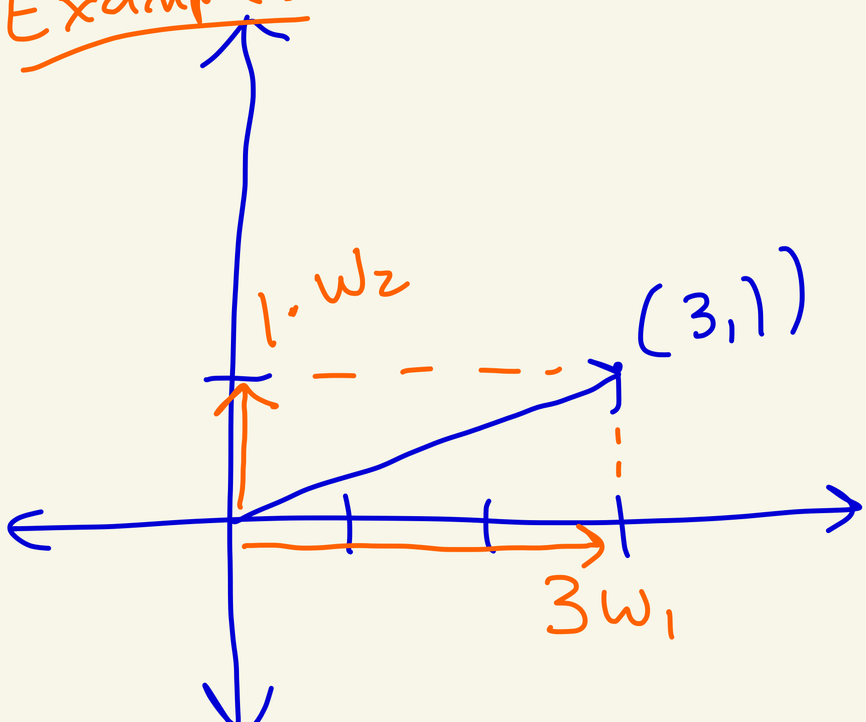
Let $w_1 = (1,0)$, $w_2 = (0,1)$.

Then,

$$\begin{aligned}\text{span}(\{w_1, w_2\}) &= \{\alpha_1 w_1 + \alpha_2 w_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\ &= \{\alpha_1 (1,0) + \alpha_2 (0,1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\ &= \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\ &= \mathbb{R}^2\end{aligned}$$

So, $\mathbb{R}^2$ is spanned by w_1 and w_2.

Example:



$(3,1) = 3w_1 + 1 \cdot w_2$

Another way to say it:

Let $(a,b) \in \mathbb{R}^2$.

Then,

$$\begin{aligned}(a,b) &= (a,0) + (0,b) \\ &= a \cdot (1,0) + b \cdot (0,1) \\ &= a \cdot w_1 + b \cdot w_2\end{aligned}$$

Thus, $(a,b) \in \text{span}(\{(1,0),(0,1)\})$.

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $v_1 = (2,1)$, $v_2 = (-1,1)$.

Do v_1, v_2 span $\mathbb{R}^2$?

Let $(a,b) \in \mathbb{R}^2$.

The question is: Can we always solve the following equation for c_1, c_2 no matter what (a,b) is?

$$(a,b) = c_1 \underbrace{(2,1)}_{v_1} + c_2 \underbrace{(-1,1)}_{v_2}$$

The above equation is equivalent to

$$(a,b) = (2c_1 - c_2,\ c_1 + c_2)$$

This is equivalent to

$$\boxed{\begin{aligned} 2c_1 - c_2 &= a \\ c_1 + c_2 &= b \end{aligned}}$$

3 operations for Gaussian elimination

1. interchange two rows
2. multiply a row by a non-zero constant
3. Add a multiple of one row to another row.

$\boxed{2550}$

We had $\boxed{\begin{array}{c} 2c_1 - c_2 = a \\ c_1 + c_2 = b \end{array}}$

$$\left(\begin{array}{cc|c} 2 & -1 & a \\ 1 & 1 & b \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 2 & -1 & a \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & -3 & -2b+a \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{3}b \end{array}\right)$$

This gives:

$$\boxed{\begin{aligned} c_1 + c_2 &= b \qquad (1) \\ c_2 &= -\frac{1}{3}a + \frac{2}{3}b \qquad (2) \end{aligned}}$$

(2) gives $c_2 = -\frac{1}{3}a + \frac{2}{3}b$.

Sub into (1) to get

$$\begin{aligned} c_1 = b - c_2 &= b - \left(-\frac{1}{3}a + \frac{2}{3}b\right) \\ &= \frac{1}{3}a + \frac{1}{3}b. \end{aligned}$$

Thus, given any $(a,b) \in \mathbb{R}^2$ we can write

$$(a,b) = \underbrace{\left(\frac{1}{3}a + \frac{1}{3}b\right)}_{c_1} \cdot \underbrace{(2,1)}_{v_1} + \underbrace{\left(-\frac{1}{3} + \frac{2}{3}b\right)}_{c_2} \cdot \underbrace{(-1,1)}_{v_2}$$

for example,

$$(1,1) = \frac{2}{3}(2,1) + \frac{1}{3}(-1,1)$$

We showed that

$$\mathbb{R}^2 = \text{span}\{(2,1), (-1,1)\}$$

Lemma: (Hw 1 #4a)

Let V be a vector space over a field F. Let $\vec{0}$ be the zero vector of V and let 0 be the zero element of F.

Then, $0w = \vec{0}$ for all $w \in V$.

proof: We have that

$$0w \overset{F3}{=} (0+0)w \overset{V8}{=} 0w + 0w$$

We know $-(0w)$ exists in V by V4.

Thus,

$$\underbrace{-(0w)+0w}_{\vec{0}} = \underbrace{-(0w)+0w}_{\vec{0}}+0w$$

So, $\vec{0} = \underbrace{\vec{0}+0w}_{0w}$

Thus, $\vec{0} = 0w$. $\blacksquare$

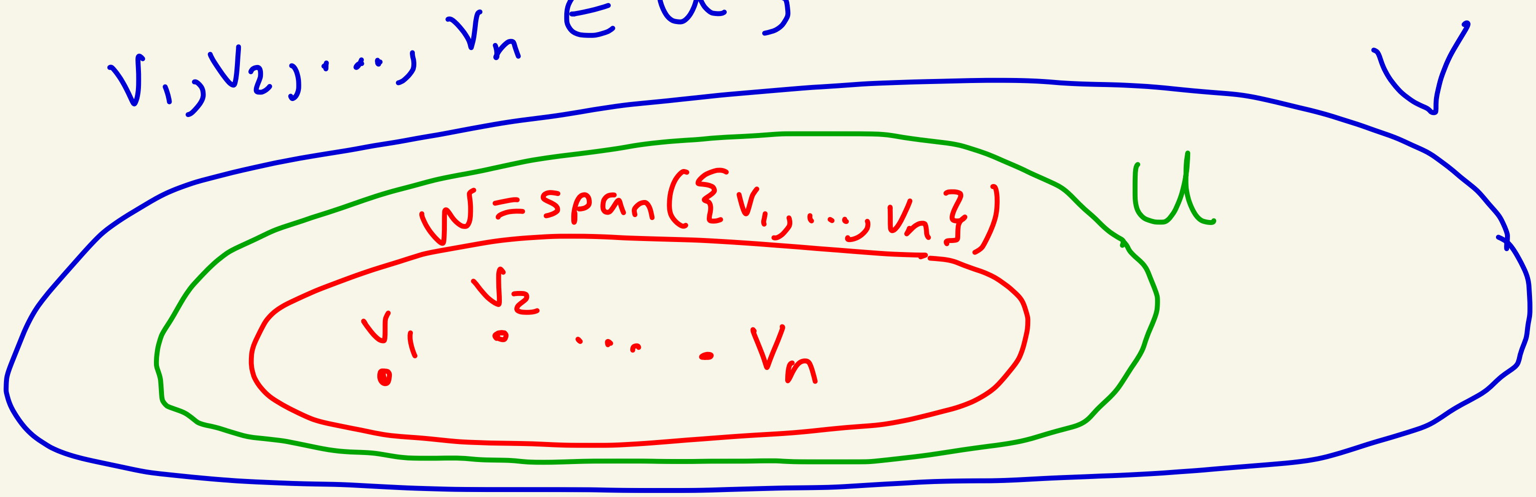
<u>Theorem:</u> Let V be a vector space over a field F.
Let $v_1, v_2, \ldots, v_n \in V$.

Let
$$W = \text{span}(\{v_1, v_2, \ldots, v_n\})$$
$$= \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \mid c_1, \ldots, c_n \in F\}$$

Then:

① W is a subspace of V.

② W is the "smallest" subspace that contains $v_1, v_2, \ldots, v_n$. That is, if U is any subspace with $v_1, v_2, \ldots, v_n \in U$, then $W \subseteq U$.



proof:

1 Let's show W is a subspace of V.

(i) If we set $c_1 = c_2 = \cdots = c_n = 0$ then we have that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n =$$

$$= 0 v_1 + 0 v_2 + \cdots + 0 v_n$$

$$\stackrel{\text{lemma}}{=} \vec{0} + \vec{0} + \cdots + \vec{0}$$

$$= \vec{0}.$$

Thus, $\vec{0} \in W$.

(ii) Let's show W is closed under $+$.

Let $w_1, w_2 \in W$.

Then, $w_1 = s_1 v_1 + s_2 v_2 + \cdots + s_n v_n$ and $w_2 = t_1 v_1 + t_2 v_2 + \cdots + t_n v_n$ where $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n \in F$.

Then,

$$w_1 + w_2 = s_1 v_1 + s_2 v_2 + \dots + s_n v_n + t_1 v_1 + t_2 v_2 + \dots + t_n v_n$$

$$= \underbrace{(s_1 + t_1)}_{\text{in } F} v_1 + \underbrace{(s_2 + t_2)}_{\text{in } F} v_2 + \dots + \underbrace{(s_n + t_n)}_{\text{in } F} v_n$$

V8

$av + bv$
$= (a+b)v$

Thus, $w_1 + w_2 \in W$, since $s_1 + t_1, s_2 + t_2, \dots, s_n + t_n \in F$.

(iii) Let's show W is closed under scalar multiplication.

Let $z \in W$ and $\alpha \in F$.

We need to show that $\alpha z \in W$.

Since $z \in W$ we know that

$$z = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

for some $c_1, c_2, \dots, c_n \in F$.

Then,

$$\alpha z = \alpha(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

$$\overset{V7}{=} \alpha(c_1 v_1) + \alpha(c_2 v_2) + \cdots + \alpha(c_n v_n)$$

$$\overset{V6}{=} \underbrace{(\alpha c_1)}_{\text{in } F \text{ (F1)}} v_1 + \underbrace{(\alpha c_2)}_{\text{in } F \text{ (F1)}} v_2 + \cdots + \underbrace{(\alpha c_n)}_{\text{in } F \text{ (F1)}} v_n$$

V7: $a(v_1+v_2) = av_1 + av_2$

V6: $(ab)w = a(bw)$

Thus, $\alpha z \in W$, because $\alpha c_1, \alpha c_2, \ldots, \alpha c_n \in F$.

By (i), (ii), (iii)

W is a subspace of V.

$\boxed{1}$

2 Let $W = \text{span}(\{v_1, v_2, \ldots, v_n\})$

Let U be a subspace of V

where $v_1, v_2, \ldots, v_n \in U$

We want to show that $W \subseteq U$.

Let $x \in W$.

Then, $x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$

where $c_1, c_2, \ldots, c_n \in F$.

Since $v_1, v_2, \ldots, v_n \in U$ and U is a subspace of V we know that $c_1 v_1, c_2 v_2, \ldots, c_n v_n \in U$.

U is closed under scalar mult.

Since $c_1 v_1, c_2 v_2, \ldots, c_n v_n \in U$ and V is a subspace of V we know that $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \in U$.

U is closed under $+$

Thus, $x \in U$.

So, $W \subseteq V$. $\boxed{2}$

<u>Def:</u> Let V be a vector space over a field F.
Let $v_1, v_2, \ldots, v_n \in V$.
We say that $v_1, v_2, \ldots, v_n$ are <u>linearly dependent</u> if there exists $c_1, c_2, \ldots, c_n \in F$, that are not all zero, such that

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \vec{0}$$

If there are no such $c_1, c_2, \ldots, c_n$ then we say that $v_1, v_2, \ldots, v_n$ are <u>linearly independent</u>.

Ex: Let $V = \mathbb{R}^3$ and $F = \mathbb{R}$.

Let $v_1 = (1, 0, 1)$

$v_2 = (-1, 2, 1)$

$v_3 = (0, 2, 2)$

Are v_1, v_2, v_3 linearly dependent or linearly independent?

We want to see what the solutions are to

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

which is

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} c_1 \\ 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ 2c_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2c_3 \\ 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} c_1 - c_2 \\ 2c_2 + 2c_3 \\ c_1 + c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} c_1 - c_2 \quad &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \end{aligned}}$$

Let's solve the system:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

leading variables c_1, c_2

free variable c_3

We get:

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \quad (1) \\ c_2 + c_3 &= 0 \quad (2) \end{aligned}}$$

solve for leading variables

$$\boxed{\begin{aligned} c_1 &= c_2 \quad (1) \\ c_2 &= -c_3 \quad (2) \end{aligned}}$$

Give free variables new name.

Let $\boxed{c_3 = t}$.

Solve (1) & (2) by back substitution.

(2) gives $\boxed{c_2 = -c_3 = -t}$

(1) gives $\boxed{c_1 = c_2 = -t}$

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Thus, the solutions to

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

are:

$$\boxed{\begin{aligned} c_1 &= -t \\ c_2 &= -t \\ c_3 &= t \end{aligned} \quad \text{where } t \text{ is any real number}}$$

$F = \mathbb{R}$

Thus,

$$-t v_1 - t v_2 + t v_3 = \vec{0}$$

for any $t \in \mathbb{R}$.

For example if $t = 1$, then

$$\boxed{-v_1 - v_2 + v_3 = \vec{0}}$$

dependency equation for v_1, v_2, v_3

$v_3 = v_1 + v_2$

Thus, v_1, v_2, v_3 are linearly dependent.

Ex: Let $V = P_2(\mathbb{R})$ and $F = \mathbb{R}$.

Let $w_1 = -3 + 4x^2$

$w_2 = 5 - x + 2x^2$

$w_3 = 1 + x + 3x^2$

Are w_1, w_2, w_3 linearly dependent or linearly independent?

Consider the equation

$$c_1 w_1 + c_2 w_2 + c_3 w_3 = \vec{0}$$

This becomes

$$c_1(-3+4x^2) + c_2(5-x+2x^2) + c_3(1+x+3x^2) = 0 + 0x + 0x^2$$

This is equivalent to

$$(-3c_1 + 5c_2 + c_3) + (-c_2 + c_3)x + (4c_1 + 2c_2 + 3c_3)x^2 = 0 + 0x + 0x^2$$

Thus, we get

$$\boxed{\begin{aligned} -3c_1 + 5c_2 + c_3 &= 0 \\ -c_2 + c_3 &= 0 \\ 4c_1 + 2c_2 + 3c_3 &= 0 \end{aligned}}$$

Solving we get

$$\left(\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -\frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -\frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{26}{3} & \frac{13}{3} & 0 \end{array}\right)$$

$$\xrightarrow[3R_3 \to R_3]{-R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 26 & 13 & 0 \end{array}\right)$$

$$\xrightarrow{-26R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 39 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{39}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} c_1 - \tfrac{5}{3}c_2 - \tfrac{1}{3}c_3 &= 0 \\ c_2 - c_3 &= 0 \\ c_3 &= 0 \end{aligned}}$$

leading variables c_1, c_2, c_3

no free variables

Solve for leading variables:

$$\boxed{\begin{aligned} c_1 &= \tfrac{5}{3}c_2 + \tfrac{1}{3}c_3 && (1) \\ c_2 &= c_3 && (2) \\ c_3 &= 0 && (3) \end{aligned}}$$

Back substitute.

(3) gives $\boxed{c_3 = 0}$

(2) gives $\boxed{c_2 = c_3 = 0}$

(1) gives $\boxed{c_1 = \frac{5}{3}c_2 + \frac{1}{3}c_3 = \frac{5}{3}(0) + \frac{1}{3}(0) = 0}$

Thus the only solution to

$$c_1 w_1 + c_2 w_2 + c_3 w_3 = \vec{0}$$

is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Thus, w_1, w_2, w_3 are linearly independent.

Summary:

You can always write

$$0 \cdot v_1 + 0 v_2 + \dots + 0 v_n = \vec{0}$$

If this is the only solution to

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}$$

then $v_1, v_2, \dots, v_n$ are linearly independent.

If there are more solutions then just the zero solution above then $v_1, v_2, \dots, v_n$ are linearly dependent.

<u>Def:</u> Let V be a vector space over a field F.
Let $v_1, v_2, \ldots, v_n \in V$.
We say that $v_1, v_2, \ldots, v_n$ form a <u>basis</u> for V if

① $\text{span}(\{v_1, v_2, \ldots, v_n\}) = V$

and ② $v_1, v_2, \ldots, v_n$ are linearly independent.

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Claim: v_1, v_2 is a basis for $V = \mathbb{R}^2$

proof:

① Last class we showed that $\text{Span}(\{v_1, v_2\}) = \mathbb{R}^2$.

② Let's show that v_1, v_2 are linearly independent.

Suppose $c_1 v_1 + c_2 v_2 = \vec{0}$

That is, $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So, $c_1 = 0$, $c_2 = 0$ is the only solution to $c_1 v_1 + c_2 v_2 = \vec{0}$.

Thus, v_1, v_2 are lin. ind.

By ① and ②, v_1, v_2 are a basis for $V = \mathbb{R}^2$ over $F = \mathbb{R}$. $\blacksquare$

Ex: Let

$$V = M_{2,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a,b,c,d \in \mathbb{R} \right\}$$

and $F = \mathbb{R}$.

Let

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $\beta = \{v_1, v_2, v_3, v_4\}$

Claim: β is a basis for $M_{2,2}(\mathbb{R})$

proof of claim:

① Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathbb{R})$.

Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{span}(\beta)$. So, β spans $M_{2,2}(\mathbb{R})$

② Suppose

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \vec{0}$$

This becomes

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Which becomes

$$\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Which gives

$$c_1 = 0, \; c_2 = 0, \; c_3 = 0, \; c_4 = 0.$$

Thus, v_1, v_2, v_3, v_4 are lin. ind.

By ① and ②, $\beta = \{v_1, v_2, v_3, v_4\}$ form a basis for $V = M_{2,2}(\mathbb{R})$ over $F = \mathbb{R}$. ▨

Theorem: Let V be a vector space over a field F.
Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a subset of V.
Then β is a basis for V if and only if every vector $x \in V$ can be expressed uniquely in the form
$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$
where $c_1, c_2, \ldots, c_n \in F$.

Proof:

($\Leftarrow$) Suppose every vector $x \in V$ can be written uniquely in the form $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$, where $c_i \in F$.

We want to show that β is a basis for V.

Since every $x \in V$ is of the form $x = c_1 v_1 + \dots + c_n v_n$ we know that

$$V = \operatorname{span}(\{v_1, \dots, v_n\}) = \operatorname{span}(\beta).$$

We now need to show that $v_1, v_2, \dots, v_n$ are lin. ind.

Suppose we want to solve

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}.$$

$\boxed{x = \vec{0}}$

We know we have

$$0v_1 + 0v_2 + \dots + 0v_n = \vec{0}$$

By our initial assumption with $x = \vec{0}$ this must be the only solution to $c_1v_1 + c_2v_2 + \dots + c_nv_n = \vec{0}$.

Thus, $v_1, v_2, \dots, v_n$ are linearly independent.

So, $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis.

($\Rightarrow$) Let β be a basis for V.

Pick some $x \in V$.

Since β is a basis for V, β spans V.

Thus, there exist $c_1, c_2, \dots, c_n \in F$ where $x = c_1v_1 + c_2v_2 + \dots + c_nv_n$. $(*)$

Let's show this expression is unique.

Suppose we also had

$$x = c_1' v_1 + c_2' v_2 + \cdots + c_n' v_n \qquad (**)$$

for some $c_1', c_2', \ldots, c_n' \in F$.

Computing $(*) - (**)$ we get

$$\vec{0} = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_n - c_n')v_n$$

Since $v_1, v_2, \ldots, v_n$ are lin. ind. we have $c_1 - c_1' = 0$, $c_2 - c_2' = 0$, $\ldots$, $c_n - c_n' = 0$.

Thus, $c_1 = c_1'$, $c_2 = c_2'$, $\ldots$, $c_n = c_n'$.

So, x can be written uniquely in the form $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$.

$\blacksquare$

$\boxed{\text{Notation for the next Theorem}}$

Consider the system

$$\boxed{\begin{aligned} 10x_1 - 3x_2 + \tfrac{1}{3}x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}} \qquad (*)$$

Let

$$\boxed{\begin{aligned} A_1 &= (10, -3, \tfrac{1}{3}) \\ A_2 &= (0, 5, -1) \\ A_3 &= (-1, 1, 0) \\ X &= (x_1, x_2, x_3) \end{aligned}}$$

Then $(*)$ can be rewritten as

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ A_3 \cdot X &= 0 \end{aligned}} \qquad \text{Same as } (*)$$

Adding $\frac{1}{10} *$ (row 1) to (row 3)

$$\boxed{\begin{aligned} 10x_1 - 3x_2 + \tfrac{1}{3}x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ \tfrac{7}{10}x_2 + \tfrac{1}{30}x_3 &= 0 \end{aligned}}$$

Which can be represented by

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ \left(\tfrac{1}{10}A_1 + A_3\right) \cdot X &= 0 \end{aligned}}$$

Theorem: Let

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right] \quad (*)$$

be a system of m equations and n unknowns where $a_{ij} \in F$ where F is a field.

If $n > m$, then $(*)$ has a non-trivial solution.

[That is, there is a solution $(x_1, x_2, \ldots, x_n) \in F^n$ to $(*)$ with $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$]

<u>Proof:</u> We induct on m [the # of equations]

<u>base case:</u> Suppose $m=1$.
We also assume $n > m = 1$. So, $n \geq 2$.
So, $(*)$ becomes

$$\boxed{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0} \quad (*)$$

If $a_{11} = a_{12} = \dots = a_{1n} = 0$, then an example of a non-trivial solution would be $x_1 = x_2 = \dots = x_n = 1$.
Suppose one of the constants isn't 0.
Without loss of generality, assume $a_{11} \neq 0$.

means: the same proof will work in other situations.

Then $(*)$ becomes

$$\boxed{x_1 = -a_{11}^{-1}(a_{12}x_2 + \dots + a_{1n}x_n)}$$

Set $x_2 = x_3 = \dots = x_n = 1$ and

$$x_1 = -a_{11}^{-1}(a_{12} + \dots + a_{1n}).$$

This gives a non-trivial solution to $(*)$.

Note we definitely used $n \geq 2$ to get the non-trivial solution.

So, the base case $m=1$ is true.

Induction hypothesis

Now assume the theorem is true for any linear system of $m-1$ equations with more than $m-1$ unknowns

Suppose we have a system $(*)$ of m equations and n unknowns with $n > m > 1$.

If all the $a_{ij} = 0$, then set $x_1 = x_2 = \cdots = x_n = 1$ and we get a non-trivial solution.

Now suppose some coefficient $a_{ij} \neq 0$. By renumbering the equations and variables we may assume $a_{11} \neq 0$.

$a_{11} \neq 0$

set
$$A_1 = (a_{11}, a_{12}, \ldots, a_{1n})$$
$$A_2 = (a_{21}, a_{22}, \ldots, a_{2n})$$
$$\vdots \qquad \vdots$$
$$A_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$$
$$X = (x_1, x_2, \ldots, x_n)$$

Then $(*)$ becomes

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ &\vdots \\ A_m \cdot X &= 0 \end{aligned}} \quad (**)$$

By subtracting a multiple of the first row and adding it to the rows below it we can eliminate x_1 in rows 2 through m. We get that

$(**)$ becomes

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ (A_2 - a_{21} a_{11}^{-1} A_1) \cdot X &= 0 \\ &\vdots \\ (A_m - a_{m1} a_{11}^{-1} A_1) \cdot X &= 0 \end{aligned}}$$

no x_1 in these rows

The last equations

$$\boxed{\begin{aligned} (A_2 - a_{21}a_{11}^{-1}A_1)\cdot X &= 0 \\ &\vdots \\ (A_m - a_{m1}a_{11}^{-1}A_1)\cdot X &= 0 \end{aligned}} \qquad (***)$$

are a system of $m-1$ equations with $n-1 > m-1$ unknowns.

Thus, by the induction hypothesis we can find a solution

$$(x_2, x_3, \ldots, x_n) \neq (0, 0, \ldots, 0)$$

to $(***)$.

Now using this solution $(x_2, \dots, x_n)$ to $(\ast\ast\ast)$ we can also solve $A_1 \cdot X = 0$ by setting

$$x_1 = -a_{11}^{-1}(a_{12}x_2 + \dots + a_{1n}x_n)$$

[because $A_1 \cdot X = 0$ is $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$ and $a_{11} \neq 0$]

Set $X = (x_1, x_2, \dots, x_n)$.

We have $A_1 \cdot X = 0$.

We also have that $i \geq 2$ then

$$A_i \cdot X \underset{(\ast\ast\ast)}{=} a_{i1} a_{11}^{-1} \underbrace{A_1 \cdot X}_{0} = 0$$

Thus we have solved

$$A_1 \cdot X = 0$$
$$A_2 \cdot X = 0$$
$$\vdots$$
$$A_m \cdot X = 0.$$

with a non-trivial solution.

$\blacksquare$

<u>Theorem:</u> Let V be a vector space over a field F.
Let $v_1, v_2, \ldots, v_m \in V$ where $V = \text{span}(\{v_1, v_2, \ldots, v_m\})$.

Let $w_1, w_2, \ldots, w_n \in V$.
If $n > m$, then $w_1, w_2, \ldots, w_n$ are linearly dependent.

<u>proof:</u> Since $v_1, v_2, \ldots, v_m$ span V we can write

$$
\begin{aligned}
w_1 &= a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m \\
w_2 &= a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m \\
&\;\;\vdots \\
w_n &= a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m
\end{aligned}
$$

where $a_{ij} \in F$.

For any $c_1, c_2, \ldots, c_n \in \mathbb{F}$ we have that

$$c_1 w_1 + c_2 w_2 + \cdots + c_n w_n =$$

$$
\begin{aligned}
&= c_1 (a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m) \\
&\quad + c_2 (a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m) \\
&\quad\;\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\quad + c_n (a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m)
\end{aligned}
$$

$$
\begin{aligned}
&= (c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n}) v_1 \\
&\quad + (c_1 a_{21} + c_2 a_{22} + \cdots + c_n a_{2n}) v_2 \\
&\quad\;\vdots \qquad\qquad\qquad \vdots \\
&\quad + (c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn}) v_m
\end{aligned}
$$

From the theorem from Monday, since $n > m$ we know that

$$\boxed{\begin{aligned} c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n} &= 0 \\ c_1 a_{21} + c_2 a_{22} + \cdots + c_n a_{2n} &= 0 \\ &\vdots \\ c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn} &= 0 \end{aligned}}$$

has a non-trivial solution $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n) \neq (0, 0, \ldots, 0)$.

Plugging this solution into the previous page we will get

$$\hat{c}_1 w_1 + \hat{c}_2 w_2 + \cdots + \hat{c}_n w_n$$
$$= 0 v_1 + 0 v_2 + \cdots + 0 v_m = \vec{0}$$

Thus, $w_1, w_2, \ldots, w_n$ are lin. dep. $\blacksquare$

<u>Corollary:</u> Let V be a vector space over a field F. Suppose $\beta_1 = \{v_1, v_2, \dots, v_a\}$ and $\beta_2 = \{w_1, w_2, \dots, w_b\}$ are both bases for V. Then $a = b$.

<u>proof:</u>

Since β_1 is a basis for V we know that β_1 spans V.

If $b > a$, then by the previous theorem, β_2 would be a linearly dependent set of vectors.

But β_2 is a basis, so the β_2 is a set of linearly independent vectors.

Thus, $\boxed{b \leq a.}$

Now we show $a \le b$.

Since β_2 is a basis for V we know that β_2 spans V.

If $a > b$, then by the previous theorem, β_1 would be a linearly dependent set of vectors.

But β_1 is a basis, so the β_1 is a set of linearly independent vectors.

Thus, $\boxed{a \le b}$.

Since $b \le a$ and $a \le b$ we know that $a = b$. ∎

The previous Corollary allows us to make the following definition.

<u>Def:</u> Let V be a vector space over a field F. We say that V is <u>finite dimensional</u> if it has a basis consisting of a finite number of elements.

If V has a basis with n elements then we say that V has <u>dimension</u> n and write $\underbrace{\dim(V) = n}_{\text{some write}}$

$\dim_F(V) = n$

$V = \{\vec{0}\}$ is called the trivial vector space.

A special case is when $V = \{\vec{0}\}$.
This vector space has no basis.
We define $V = \{\vec{0}\}$ to have dimension zero,
that is $\dim(\{\vec{0}\}) = 0$.

Ex: Let F be a field and $V = F^n$ where $n \geq 1$.

Recall $V = F^n$ is a vector space over F.

We now show that $\dim(F^n) = n$

Proof: We will construct what is called the standard basis.

Let v_i be the vector with a 1 in the i-th spot and 0's everywhere else.

That is,

$$
\begin{aligned}
v_1 &= (1, 0, 0, \ldots, 0) \\
v_2 &= (0, 1, 0, \ldots, 0) \\
&\vdots \\
v_n &= (0, 0, 0, \ldots, 1)
\end{aligned}
$$

Let $\beta = \{v_1, v_2, \ldots, v_n\}$

We will now show that β is a basis for $V = F^n$ which will give us that $\dim(F^n) = n$.

① $\underline{\beta \text{ spans } V = F^n}$:

Let $x \in F^n$

Then, $x = (f_1, f_2, \ldots, f_n)$ where $f_1, f_2, \ldots, f_n \in F$.

So,

$$\begin{aligned} x &= (f_1, f_2, \ldots, f_n) \\ &= (f_1, 0, \ldots, 0) + (0, f_2, \ldots, 0) \\ &\quad + \cdots + (0, 0, \ldots, f_n) \\ &= f_1(1, 0, \ldots, 0) + f_2(0, 1, \ldots, 0) \\ &\quad + \cdots + f_n(0, 0, \ldots, 1) \\ &= f_1 v_1 + f_2 v_2 + \cdots + f_n v_n \end{aligned}$$

Thus, $x \in \text{span}(\beta)$.

Therefore, β spans $V = F^n$.

2 β is linearly independent:

Suppose

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}$$

where $c_1, c_2, \dots, c_n \in F$.

Then,

$$c_1(1, 0, \dots, 0) + c_2(0, 1, \dots, 0) + \dots + c_n(0, 0, \dots, 1) = (0, 0, \dots, 0)$$

So, $(c_1, 0, \dots, 0) + (0, c_2, \dots, 0) + \dots + (0, 0, \dots, c_n) = (0, 0, \dots, 0)$.

Ergo, $(c_1, c_2, \dots, c_n) = (0, 0, \dots, 0)$.

So, $c_1 = 0, c_2 = 0, \dots, c_n = 0$.

Thence, $v_1, v_2, \dots, v_n$ are lin. independent.

$\blacksquare$

Ex: Let $F=\mathbb{R}$ or $F=\mathbb{C}$.

Let
$$V = P_n(F) = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in F\}$$

One can show that

$$\left.\begin{aligned} v_0 &= 1 \\ v_1 &= x \\ v_2 &= x^2 \\ &\vdots \\ v_n &= x^n \end{aligned}\right\} \text{n+1 vectors}$$

is a basis for $P_n(F)$ over F.

So, $\dim(P_n(F)) = n+1$

Ex: Let F be a field and $V = M_{m,n}(F)$ be the set of $m \times n$ matrices with entries from F.

One can show that

$$\dim(M_{m,n}(F)) = m \cdot n$$

For example,

$$M_{3,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \middle| \, a,b,c,d,e,f \in \mathbb{R} \right\}$$

A basis for $M_{3,2}(\mathbb{R})$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $\dim(M_{3,2}(\mathbb{R})) = 3 \cdot 2 = 6$

S3

Theorem: Let V be a vector space over a field F. Suppose $\dim(V) = n > 0$. Then the following are true:

① Let $v_1, v_2, \ldots, v_m \in V$.

(a) If $m > n$, then $v_1, v_2, \ldots, v_m$ are linearly dependent.

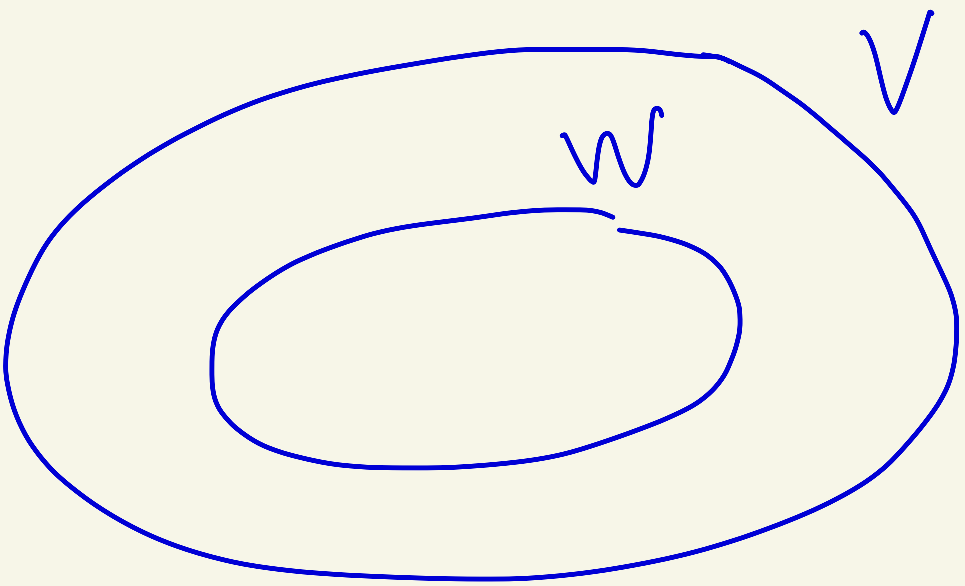
(b) If $m < n$, then $v_1, v_2, \ldots, v_m$ do not span V.

(c) If $m = n$ and $v_1, v_2, \ldots, v_m$ span V, then $v_1, v_2, \ldots, v_m$ are also linearly independent and hence form a basis for V.

(d) If $m = n$ and $v_1, v_2, \ldots, v_m$ are linearly independent, then $v_1, v_2, \ldots, v_m$ span V and hence form a basis for V.

2. Let W be a subspace of V. Then W is finite-dimensional and $\dim(W) \leq \underbrace{n}_{\dim(V)}$

Moreover, $W = V$ if and only if $\dim(W) = \dim(V)$.



<u>proof:</u> We have that $\dim(V) = n$.

① Let $v_1, v_2, \ldots, v_m \in V$.

(a) Suppose that $m > n$.

Since $\dim(V) = n$ we know that V has a basis with n vectors.

So, V is spanned by n vectors.

From a previous theorem, since $m > n$ we know that $v_1, v_2, \ldots, v_m$ are linearly dependent.

(b) Suppose $m < n$.

Let's show that $v_1, v_2, \ldots, v_m$ do not span V.

Suppose instead that $v_1, v_2, \ldots, v_m$ <u>did</u> span V.

Then from our previous results,
since $m<n$, and $v_1, v_2, \ldots, v_m$
span V, we would have
that any set of n vectors
must be linearly dependent.

But since $\dim(V)=n$ there
must be a basis for V
of size n.

So, there is a set of n vectors
in V that are linearly
independent.

Contradiction.

So, $v_1, v_2, \ldots, v_m$ do not span V.

(c) Suppose $m = n$ and $v_1, v_2, \ldots, v_m$ span V

We want to show that $v_1, v_2, \ldots, v_m$ are linearly independent.

HW 2 - #7b

Suppose $V \neq \{\vec{0}\}$ is spanned by some finite set S of vectors. Prove that some subset of S is a basis for V

Let $S = \{v_1, v_2, \ldots, v_m\}$.

By this HW problem, there is a subset S' of S that is a basis for V.

Since $\dim(V) = n$, every basis for V has n vectors in it.

So, S' has $m = n$ vectors.

Thus, $S' = S$. Thus, $S = \{v_1, v_2, \ldots, v_m\}$ is a basis for V and is thus linearly independent.

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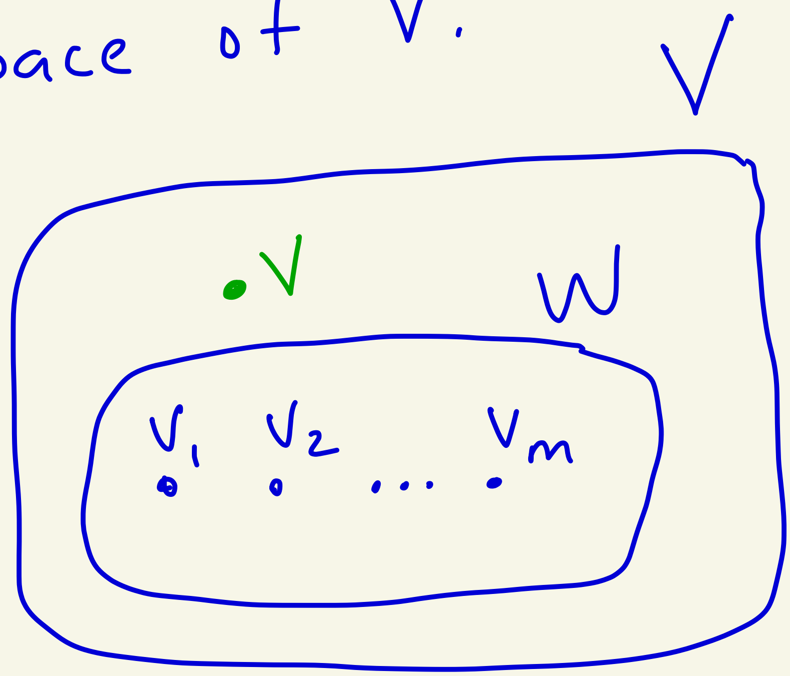
(d) Suppose $m = n = \dim(V)$ and $v_1, v_2, \ldots, v_m$ are linearly independent.

We want to show that $v_1, v_2, \ldots, v_m$ span V and hence are a basis for V.

Let $W = \mathrm{span}(\{v_1, v_2, \ldots, v_m\})$.

So W is a subspace of V.



We will now show that $W = V$.

We know $W \subseteq V$.

We need to show that $V \subseteq W$.

Let $v \in V$.

Since $\dim(V) = n = m$ we know that the $n+1 = m+1$ vectors $v_1, v_2, \ldots, v_m, v$ are linearly dependent from part (a).

Thus, there exist

$c_1, c_2, \ldots, c_m, c_{m+1} \in F$,

not all equal to zero, where

$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m + c_{m+1} v = \vec{0}$

If $c_{m+1} = 0$, then

$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = \vec{0}$

with not all $c_1, c_2, \ldots, c_m$ equalling zero.

But this would contradict the fact that $v_1, v_2, \ldots, v_m$ are linearly independent.

Thus, $c_{m+1} \neq 0$.

So, we can solve for v in

$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m + c_{m+1} v = \vec{0}$

and we get

$$v = \underbrace{c_{m+1}^{-1}}_{\substack{\text{exists} \\ \text{since } c_{m+1} \neq 0}} \left(-c_1 v_1 - c_2 v_2 - \dots - c_m v_m\right)$$

So,

$$v = \left(-c_{m+1}^{-1} c_1\right) v_1 + \left(-c_{m+1}^{-1} c_2\right) v_2 + \dots + \left(-c_{m+1}^{-1} c_m\right) v_m$$

Thus, $v \in \operatorname{span}(\{v_1, v_2, \dots, v_m\}) = W$.

So, $V = W$ and $v_1, v_2, \dots, v_m$ span V and are thus a basis for V. $\boxed{1}$

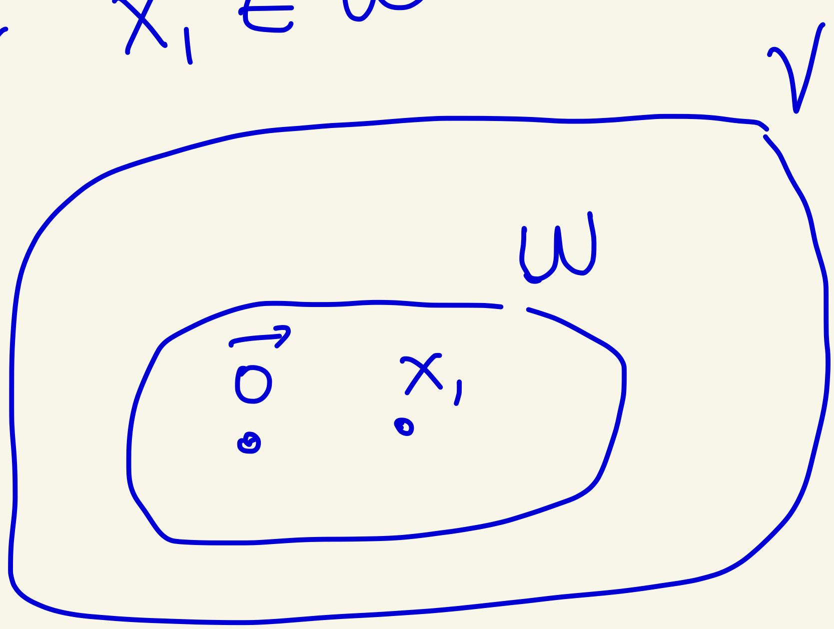
Now for part 2.

2

Let W be a subspace of V.
We first will show that W is finite-dimensional and $\dim(W) \leq n = \dim(V)$.

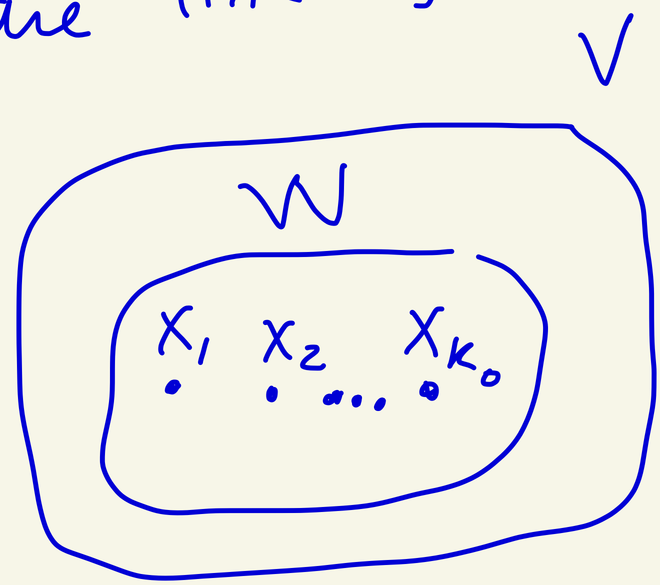
If $W = \{\vec{0}\}$, then W is finite-dimensional and $\dim(W) = 0 < n = \dim(V)$.

Now suppose $W \neq \{\vec{0}\}$.
Then there exists $x_1 \in W$ with $x_1 \neq \vec{0}$.



Then, $\{x_1\}$ is a linearly independent set of vectors.
Because if $c_1 x_1 = \vec{0}$ then $c_1 = 0$ because $x_1 \neq \vec{0}$.

Continue to add vectors from W to this set such that at each stage k, the vectors $\{x_1, x_2, \ldots, x_k\}$ are linearly independent.



Since $W \subseteq V$ and $\dim(V) = n$, by part (a), there must reach a stage $k_0 \leq n$ where $S_0 = \{x_1, x_2, \ldots, x_{k_0}\}$ is linearly independent but adding any new vector from W to S_0 will yield a linearly dependent set.

HW 2 – 7(a)

Let S be a finite set of linearly independent vectors from V and let $x \in V$ with $x \notin S$. Then $S \cup \{x\}$ is linearly dependent iff $x \in \text{span}(S)$

Let $x \in W$.

If $x \in S_0$, then $x \in \text{span}(S_0)$.

If $x \notin S_0$, then by the construction of S_0 we have that $S_0 \cup \{x\}$ is linearly dependent. So by HW 2, 7(a), $x \in \text{span}(S_0)$.

Thus, if $x \in W$, then $x \in \text{span}(S_0)$.

So, $W = \text{span}(S_0)$.

Since S_0 is a lin. ind. set, S_0 is a basis for W. Thus, $\dim(W) = k_0 \leq n = \dim(V)$.

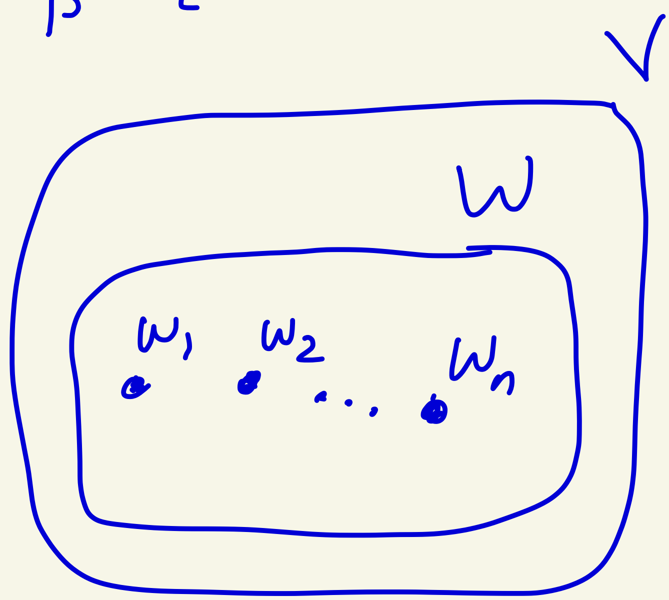
Now we show that $W = V$
iff $\dim(W) = \dim(V)$.

($\Rightarrow$) If $V = W$, then $\dim(V) = \dim(W)$.

($\Leftarrow$) Now suppose $\dim(W) = \dim(V)$.
Let's show that $W = V$.

Then W has a basis of $n = \dim(V)$
elements, call it $\beta = \{w_1, w_2, \ldots, w_n\}$

So, $W = \text{span}(\beta)$.



By part 1(d), since
β is a set of
n vectors that
are linearly independent and
$n = \dim(V)$, they must span
V also!

So, β is a basis for V.
Thus, $W = \text{span}(\beta) = V$. $\blacksquare$