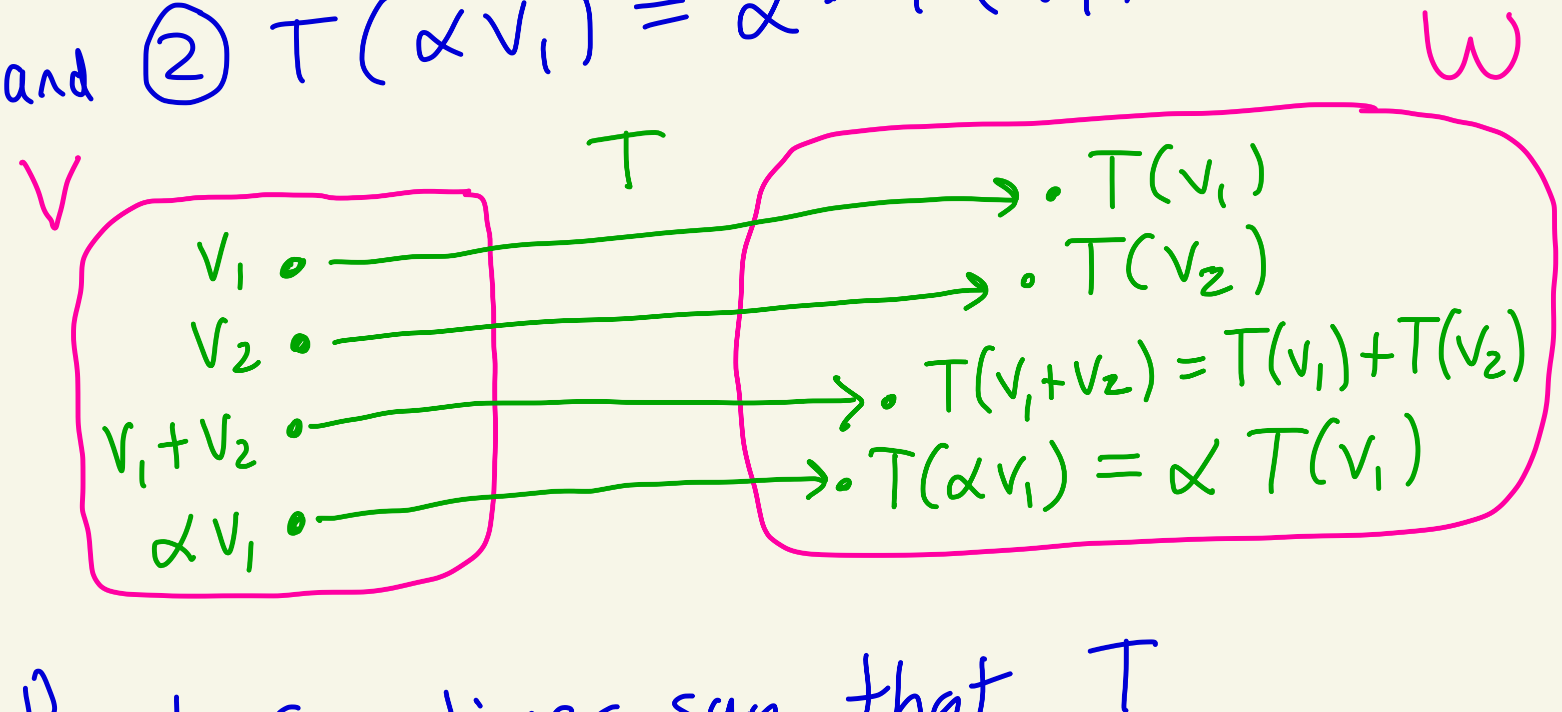
Topic 3 – Linear Transformations

<u>Def:</u> Let V and W be vector spaces over a field F. Let $T: V \to W$ be a function between them. We say that T is a <u>linear transformation</u> if for every $v_1, v_2 \in V$ and $\alpha \in F$ we have that

1. $T(v_1 + v_2) = T(v_1) + T(v_2)$

and

2. $T(\alpha v_1) = \alpha \cdot T(v_1)$



People sometimes say that T "preserves" vector addition and scalar multiplication

②

You can condense ① and ② into one condition:

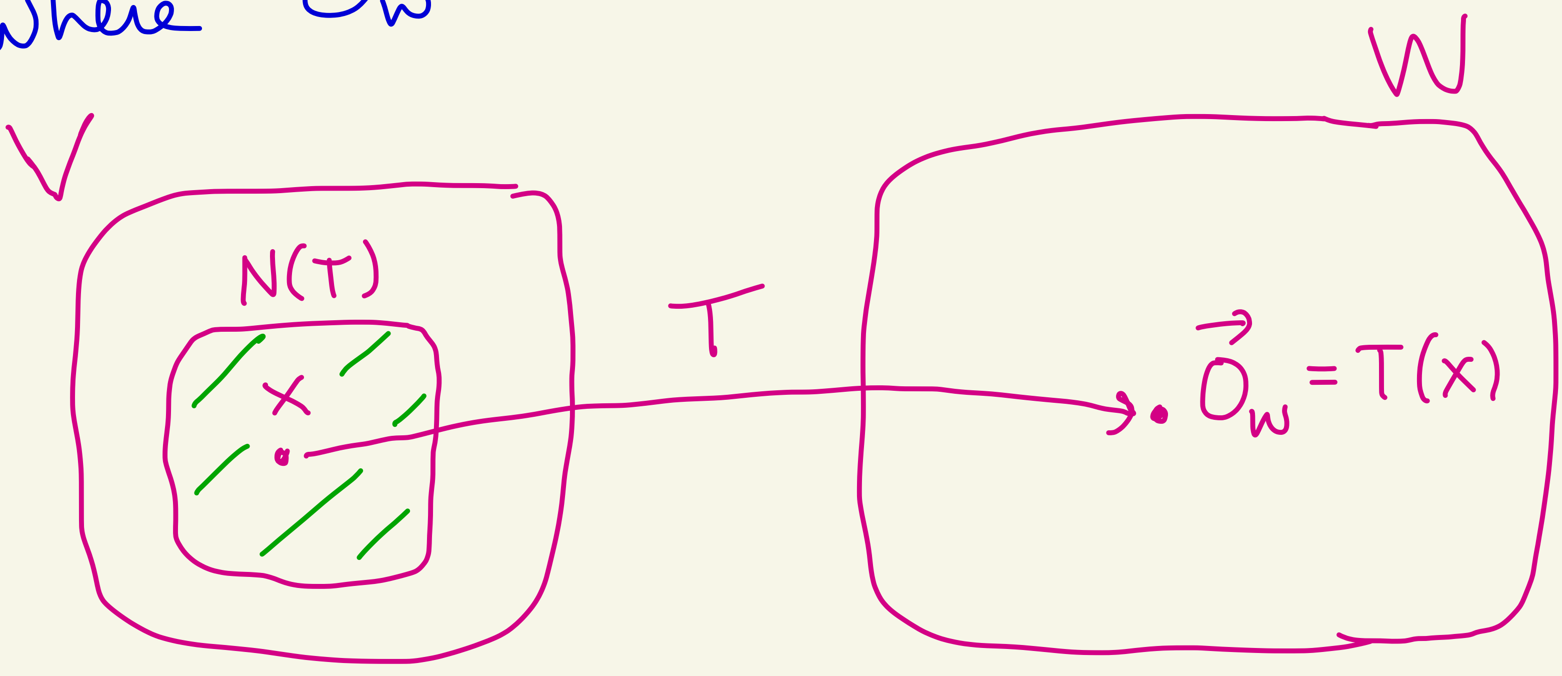
$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$$

for all $v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in F$

We define the <u>nullspace</u> (or <u>kernel</u>) of T to be

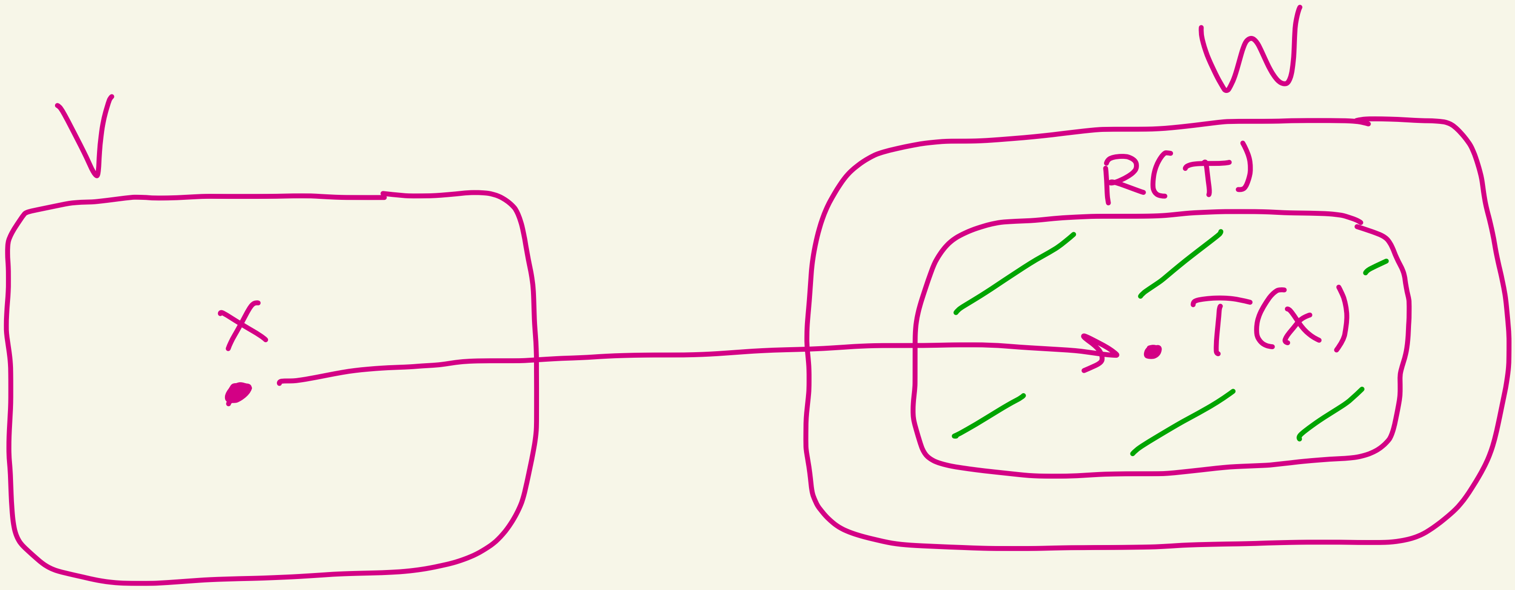
$$N(T) = \{ x \in V \mid T(x) = \vec{0}_W \}$$

where $\vec{0}_W$ is the zero vector of W.



We define the <u>range</u> (or <u>image</u>) of T to be

$$R(T) = \{T(x) \mid x \in V\}$$



<u>Comment:</u> We will show later that $N(T)$ is a subspace of V and $R(T)$ is a subspace of W

If $N(T)$ is finite-dimensional then we call the dimension of $N(T)$ the <u>nullity of T</u> and write

$$\text{nullity}(T) = \dim(N(T))$$

If $R(T)$ is finite-dimensional then we call the dimension of $R(T)$ the <u>rank of T</u> and write

$$\text{rank}(T) = \dim(R(T))$$

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Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x,y,z) = (x,y)$

Here $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $F = \mathbb{R}$.

For example,

$$T(1, \pi, 10) = (1, \pi)$$

$$T(-1, \tfrac{1}{2}, 3) = (-1, \tfrac{1}{2})$$

T is a linear transformation:

proof: Let $v_1, v_2 \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

Then, $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ where $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$.

(1) Then,

$$T(v_1 + v_2)$$

$$= T\big((x_1, y_1, z_1) + (x_2, y_2, z_2)\big)$$

$$= T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= (x_1 + x_2,\ y_1 + y_2)$$

$$= (x_1, y_1) + (x_2, y_2)$$

$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

$$= T(v_1) + T(v_2)$$

2 We also have that

$$T(\alpha v_1) = T(\alpha(x_1, y_1, z_1))$$

$$= T(\alpha x_1, \alpha y_1, \alpha z_1)$$

$$= (\alpha x_1, \alpha y_1)$$

$$= \alpha \cdot (x_1, y_1)$$

$$= \alpha \cdot T(x_1, y_1, z_1)$$

$$= \alpha \cdot T(v_1)$$

$\blacksquare$

$\boxed{\text{Nullspace of } T}$:

$$N(T) = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{(x,y) = (0,0)}_{x=0 \text{ and } y=0}\}$$

$$= \boxed{\{(0,0,z) \mid z \in \mathbb{R}\}} \leftarrow \boxed{N(T)}$$

$$= \{z \cdot (0,0,1) \mid z \in \mathbb{R}\}$$

$$= \text{span}(\{(0,0,1)\})$$

Let $\beta = \{(0,0,1)\}$.

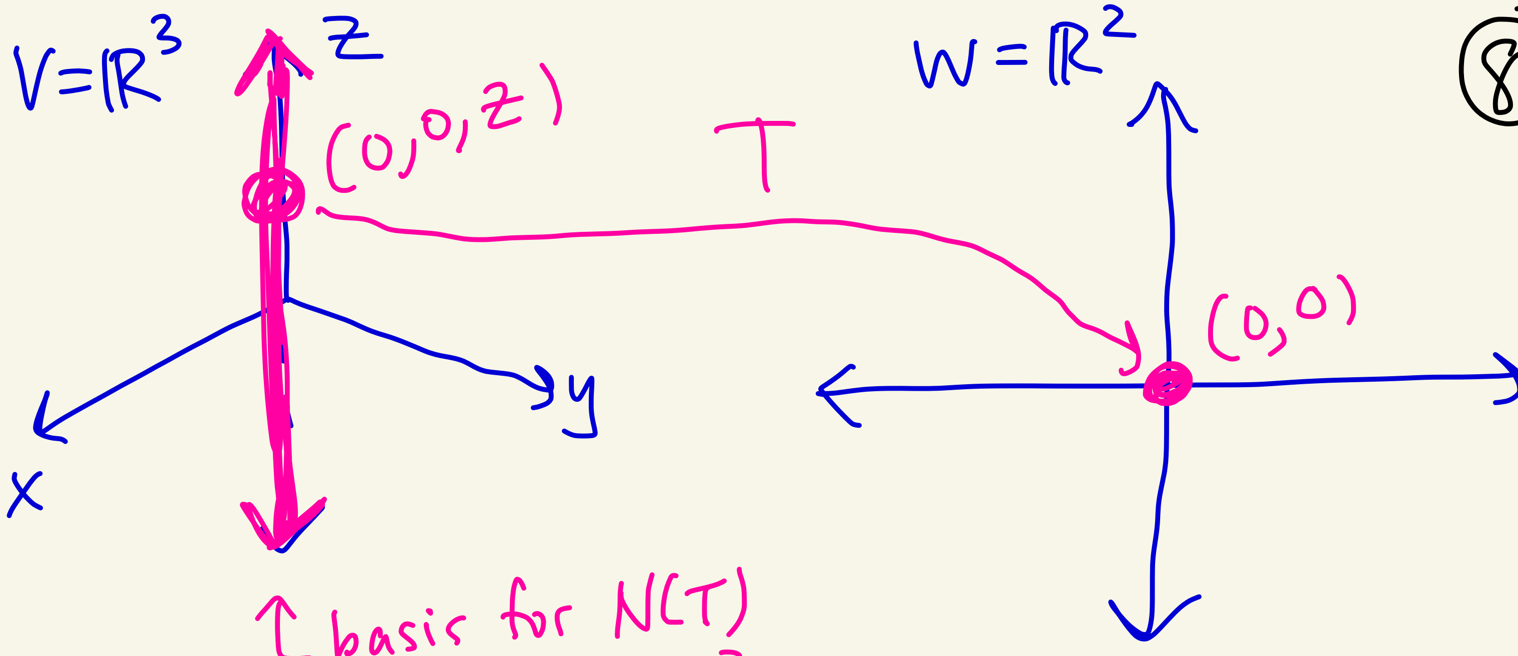
Then β spans $N(T)$.

By HW 2 #6 since β consists of one non-zero vector, β is a linearly independent set.

So, β is a basis for $N(T)$ and

$$\text{nullity}(T) = \dim(N(T)) = \left(\begin{array}{c}\#\text{ elements in} \\ \text{basis } \beta\end{array}\right)$$

$$= 1$$

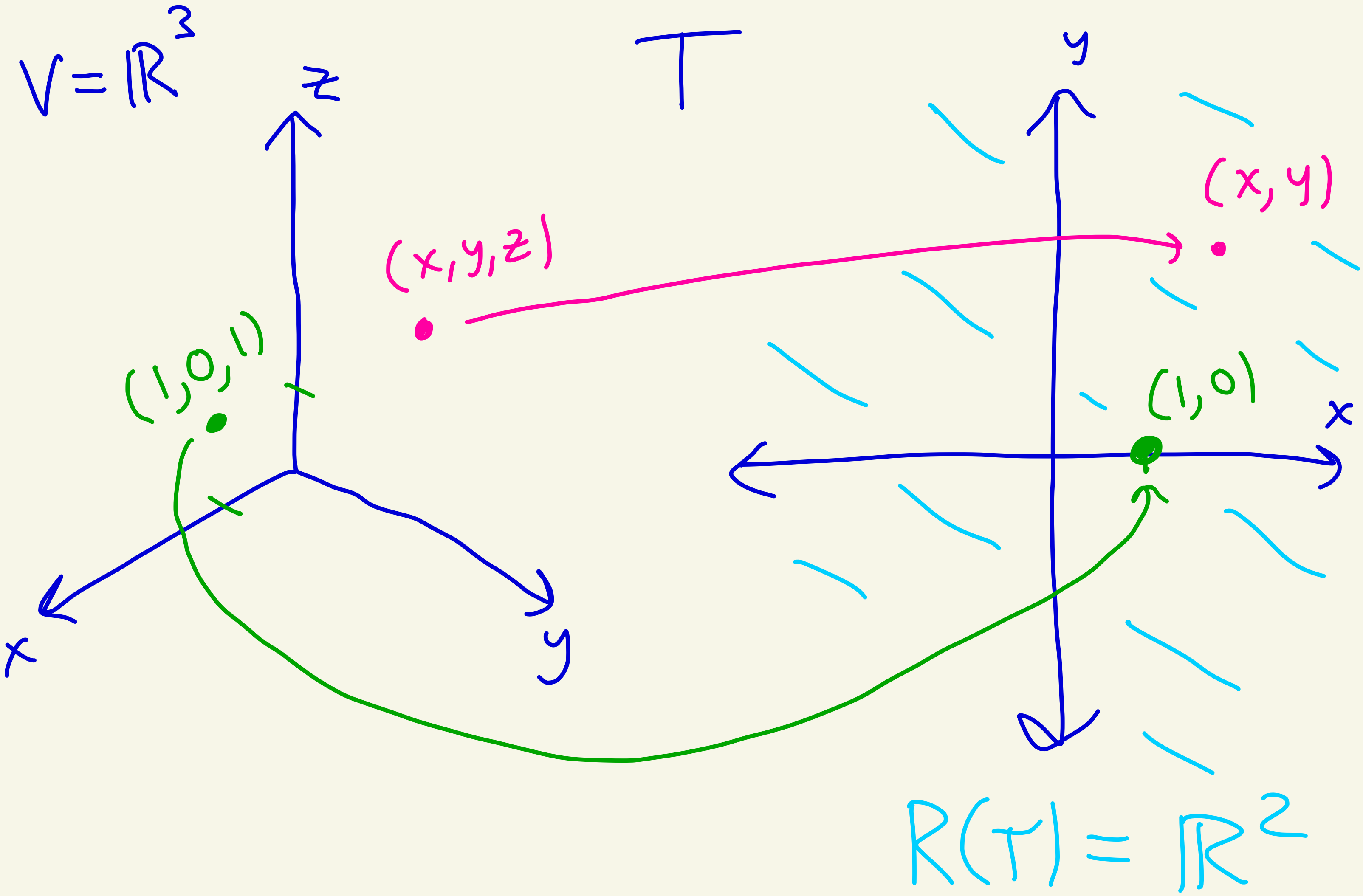


basis for $N(T)$ is $\{(0,0,1)\}$

Range of T

$$R(T) = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$$
$$= \{(x,y) \mid (x,y,z) \in \mathbb{R}^3\}$$
$$= \{(x,y) \mid x,y \in \mathbb{R}\}$$
$$= \mathbb{R}^2$$

Thus, $\operatorname{rank}(T) = \dim(R(T))$
$$= \dim(\mathbb{R}^2) = 2$$



Note:

$$\dim(V) = \text{nullity}(T) + \text{rank}(T)$$

$$3 = 1 + 2$$

Ex: Let $n \geq 1$ be fixed and

$$T : \underbrace{P_n(\mathbb{R})}_{\substack{\text{polys} \\ \text{of degree} \\ \leq n}} \longrightarrow \underbrace{P_{n-1}(\mathbb{R})}_{\substack{\text{polys} \\ \text{of degree} \\ \leq n-1}}$$

where $T(f) = f'$.
Here f' is the derivative of the polynomial f.

T is a linear transformation:

Let $f_1, f_2 \in P_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$.
Then,

$$T(f_1 + f_2) = (f_1 + f_2)' = f_1' + f_2' = T(f_1) + T(f_2)$$

property of derivatives

and

$$T(\alpha f_1) = (\alpha f_1)' = \alpha f_1' = \alpha T(f_1)$$

$\blacksquare$

Nullspace of T:

$N(T) =$

$= \{a_0 + a_1 x + \dots + a_n x^n \mid T(a_0 + a_1 x + \dots + a_n x^n) = \vec{0}\}$

$= \{a_0 + a_1 x + \dots + a_n x^n \mid \underbrace{a_1}_{0} + \underbrace{2a_2}_{0} x + \dots + \underbrace{na_n}_{0} x^{n-1} = \vec{0}\}$

$= \{a_0 + a_1 x + \dots + a_n x^n \mid a_1 = a_2 = \dots = a_n = 0\}$

$= \{a_0 \mid a_0 \in \mathbb{R}\}$ ← constant polynomials

$= \{a_0 \cdot 1 \mid a_0 \in \mathbb{R}\}$

$= \text{span}(\{1\})$

Let $\beta = \{1\}$. Then β spans $N(T)$.

Since β consists of one non-zero vector by HW 2 #6, β is a lin. ind. set.

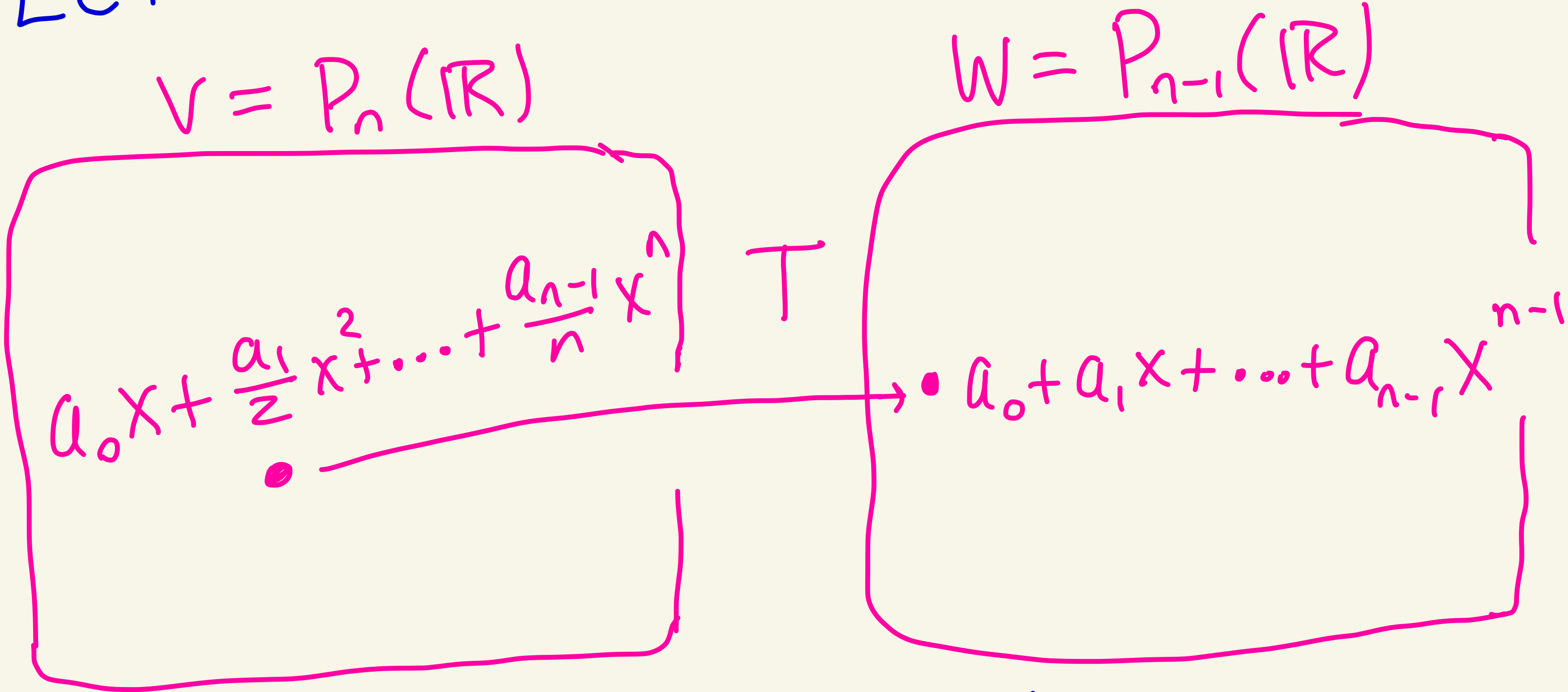
Thus, β is a basis for $N(T)$.

So, $\text{nullity}(T) = \left(\begin{array}{l}\text{\# elements in} \\ \text{basis } \beta\end{array}\right) = 1.$

Range of T:

I claim that T is onto.
That is, $R(T) = P_{n-1}(\mathbb{R})$.

Let $a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in P_{n-1}(\mathbb{R})$.



Integrate and notice that

$$a_0 x + \frac{a_1}{2} x^2 + \dots + \frac{a_{n-1}}{n} x^n \in P_n(\mathbb{R})$$

and

$$T\left(a_0 x + \frac{a_1}{2} x^2 + \dots + \frac{a_{n-1}}{n} x^n\right)$$

$$= a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$$

Thus, T is onto $P_{n-1}(\mathbb{R}) = W$.

Thus,

$R(T) = P_{n-1}(\mathbb{R})$.

So, $\text{rank}(T) = \dim(P_{n-1}(\mathbb{R}))$

$= (n-1)+1 = n$.

Note:

$$\dim(\underbrace{P_n(\mathbb{R})}_{V}) = \text{nullity}(T) + \text{rank}(T)$$

$$n+1 \quad = \quad 1 \quad + \quad n$$

Another way to make a linear transformation is by matrix multiplication

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<u>Def:</u> Let F be a field. Let A be an $m \times n$ matrix with coefficients from F. We can construct a linear transformation

$$L_A : F^n \longrightarrow F^m$$

where $L_A(x) = Ax$ for any $x \in F^n$.

[Here Ax is matrix multiplication]

L_A is called the <u>left-multiplication by A transformation</u>.

$$\underbrace{A}_{m \times n} \; \underbrace{x}_{n \times 1}$$

result is $m \times 1$

<u>Note:</u> L_A above is a linear transformation because if $x, y \in F^n$ and $\alpha, \beta \in F$ then

$$\begin{aligned} L_A(\alpha x + \beta y) &= A(\alpha x + \beta y) \\ &= A(\alpha x) + A(\beta y) \\ &= \alpha A x + \beta A y \\ &= \alpha L_A(x) + \beta L_A(y) \end{aligned}$$

property of matrix multiplication

$\blacksquare$

Ex: Let $F = \mathbb{C}$.

Let

$$A = \begin{pmatrix} i & 1+i & -3-5i \\ 0 & 1 & -1-i \end{pmatrix}$$

be in $M_{2\times 3}(\mathbb{C})$.

$\boxed{i^2 = -1}$

Then,

$$L_A : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$$

where

$$L_A\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} i & 1+i & -3-5i \\ 0 & 1 & -1-i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

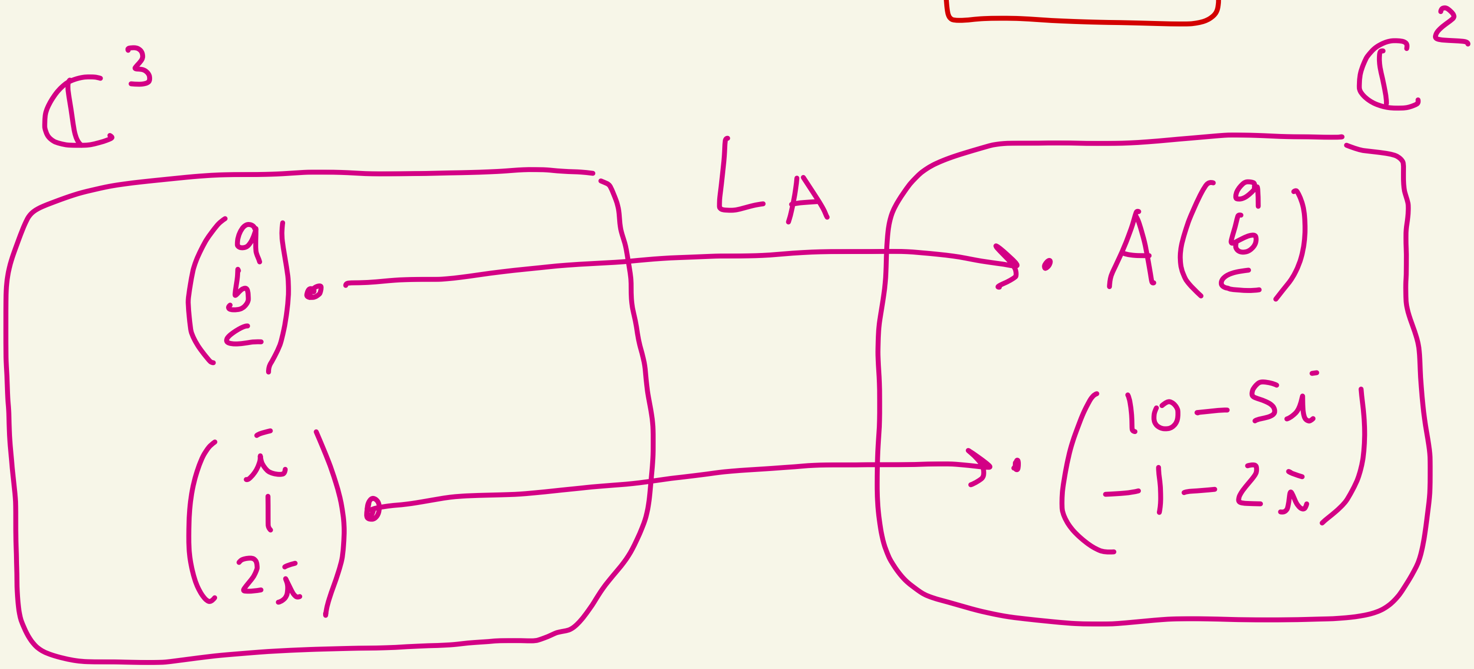
For example,

$$L_A\begin{pmatrix} i \\ 1 \\ 2i \end{pmatrix} = \begin{pmatrix} i & 1+i & -3-5i \\ 0 & 1 & -1-i \end{pmatrix}\begin{pmatrix} i \\ 1 \\ 2i \end{pmatrix}$$

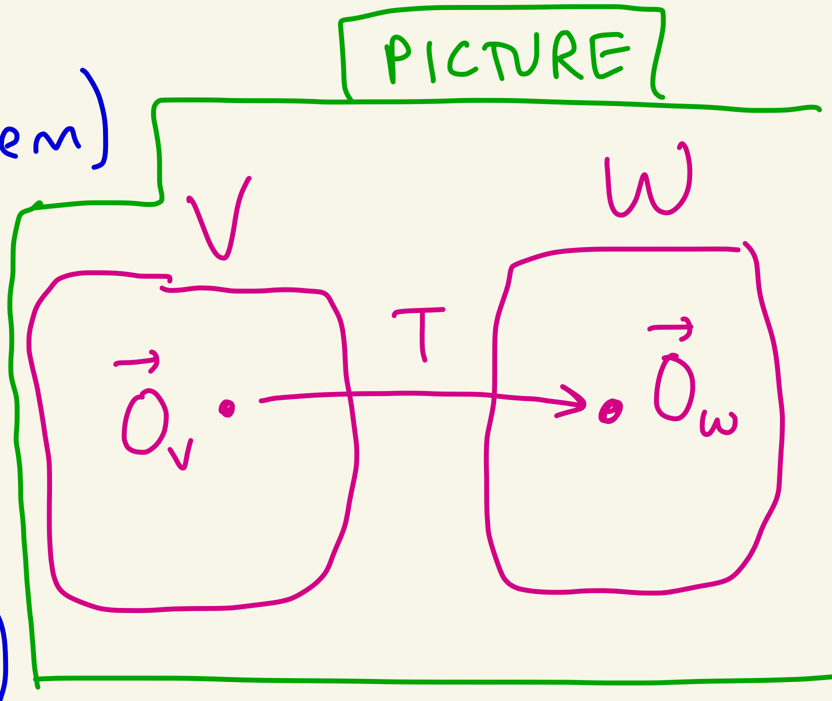
$$= \begin{pmatrix} (i)(i) + (1+i)(1) + (-3-5i)(2i) \\ (0)(i) + (1)(1) + (-1-i)(2i) \end{pmatrix}$$

$$= \begin{pmatrix} i^2 + 1 + i - 6i - 10i^2 \\ 0 + 1 - 2i + 2i^2 \end{pmatrix} = \begin{pmatrix} 10-5i \\ -1-2i \end{pmatrix}$$

$\boxed{i^2 = -1}$



Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W respectively. Then, $T(\vec{0}_V) = \vec{0}_W$.



proof: (HW problem)

We have that

$$T(\vec{0}_V) = T(\vec{0}_V + \vec{0}_V)$$

$$= T(\vec{0}_V) + T(\vec{0}_V)$$

T is linear

Thus, $T(\vec{0}_V) = T(\vec{0}_V) + T(\vec{0}_V)$ in W.

Add the additive inverse $-T(\vec{0}_V)$ to both sides to get that

$$\underbrace{-T(\vec{O}_v)+T(\vec{O}_v)}_{\vec{O}_w} = -\underbrace{T(\vec{O}_v)+T(\vec{O}_v)}_{\vec{O}_w}+T(\vec{O}_v)$$

So,
$$\vec{O}_w = \underbrace{\vec{O}_w + T(\vec{O}_v)}_{T(\vec{O}_v)}$$

Thus, $T(\vec{O}_v) = \vec{O}_w$. $\blacksquare$

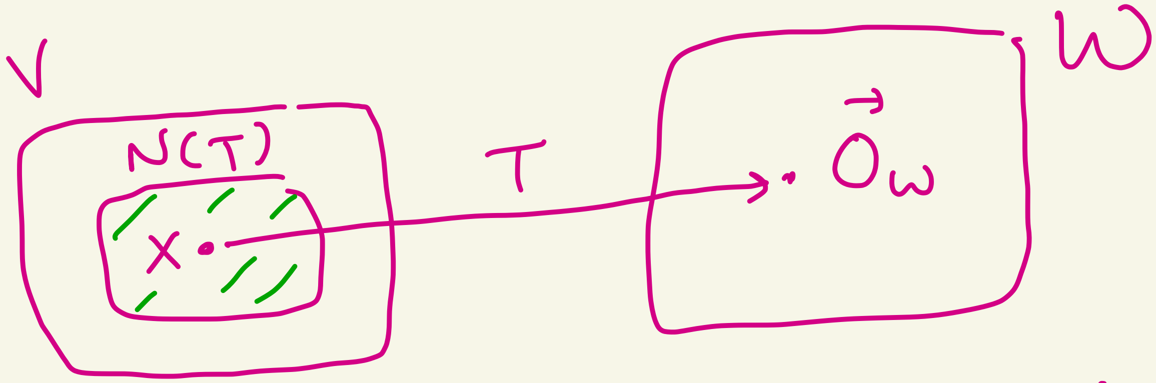
Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

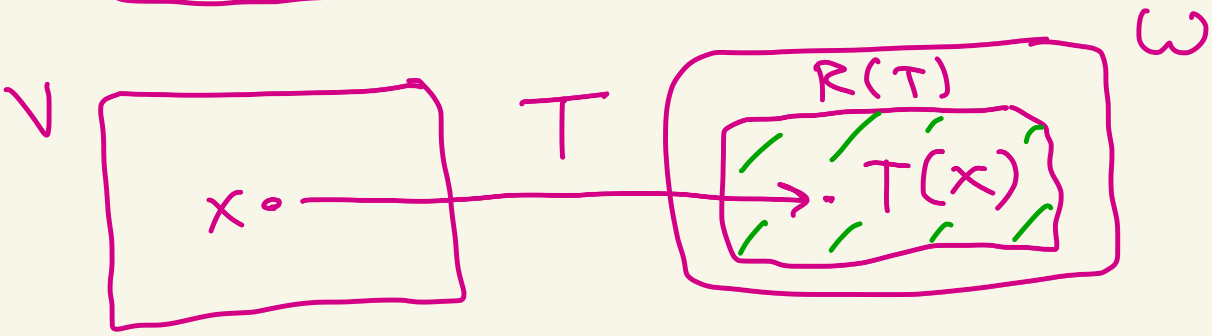
Then:

① $N(T) = \{ x \in V \mid T(x) = \vec{0}_W \}$ is a subspace of V

and

② $R(T) = \{ T(x) \mid x \in V \}$ is a subspace of W.





proof: Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W.

① Let's show that $N(T)$ is a subspace of V.

(i) By the previous theorem today we know that $T(\vec{0}_V) = \vec{0}_W$. This tells us that $\vec{0}_V \in N(T)$.

(ii) Let's show $N(T)$ is closed under $+$.

Let $x, y \in N(T)$.

Then, $T(x) = \vec{0}_W$ and $T(y) = \vec{0}_W$

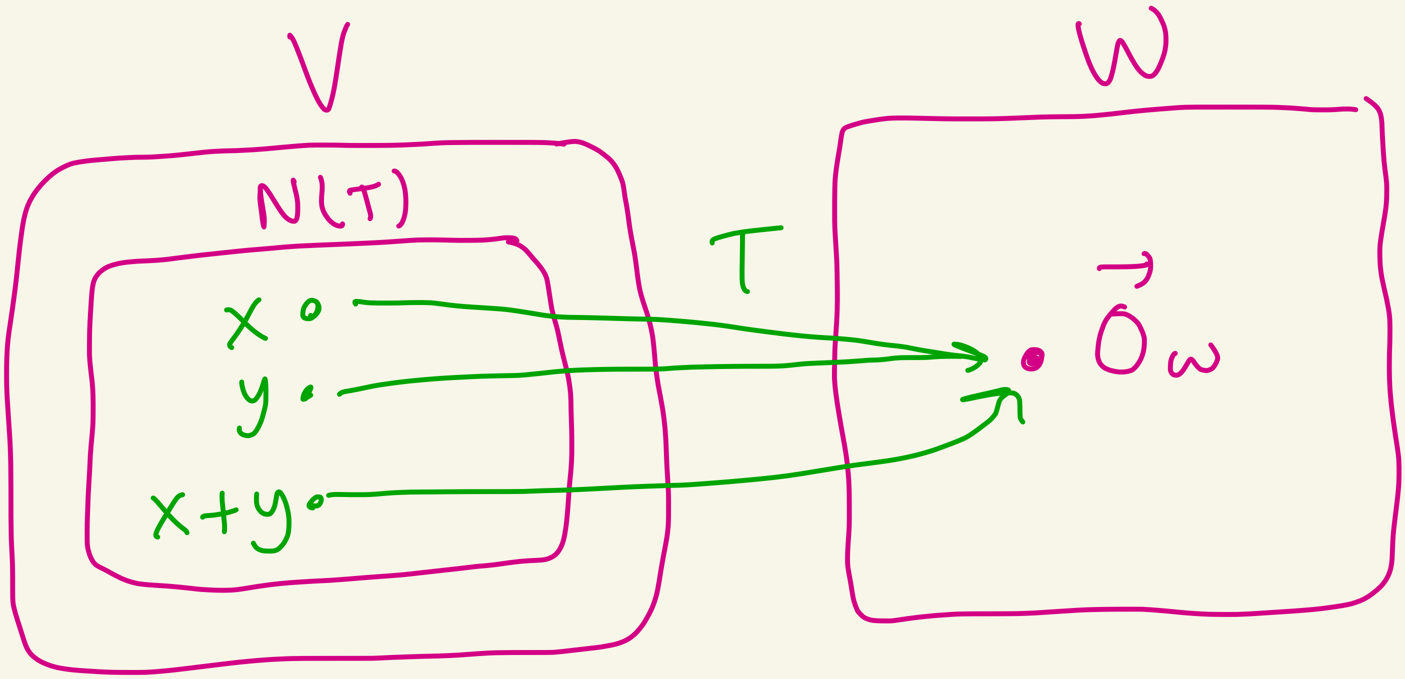
So, $T(x+y) = T(x) + T(y)$

$= \vec{0}_W + \vec{0}_W = \vec{0}_W$

Since T is linear

Thus, $T(x+y) = \vec{0}_W$.

So, $x+y \in N(T)$.



(iii) Let's show $N(T)$ is closed under scalar mult.

Let $z \in N(T)$ and $\alpha \in F$.

Since $z \in N(T)$, we know that $T(z) = \vec{0}_w$.

Thus,

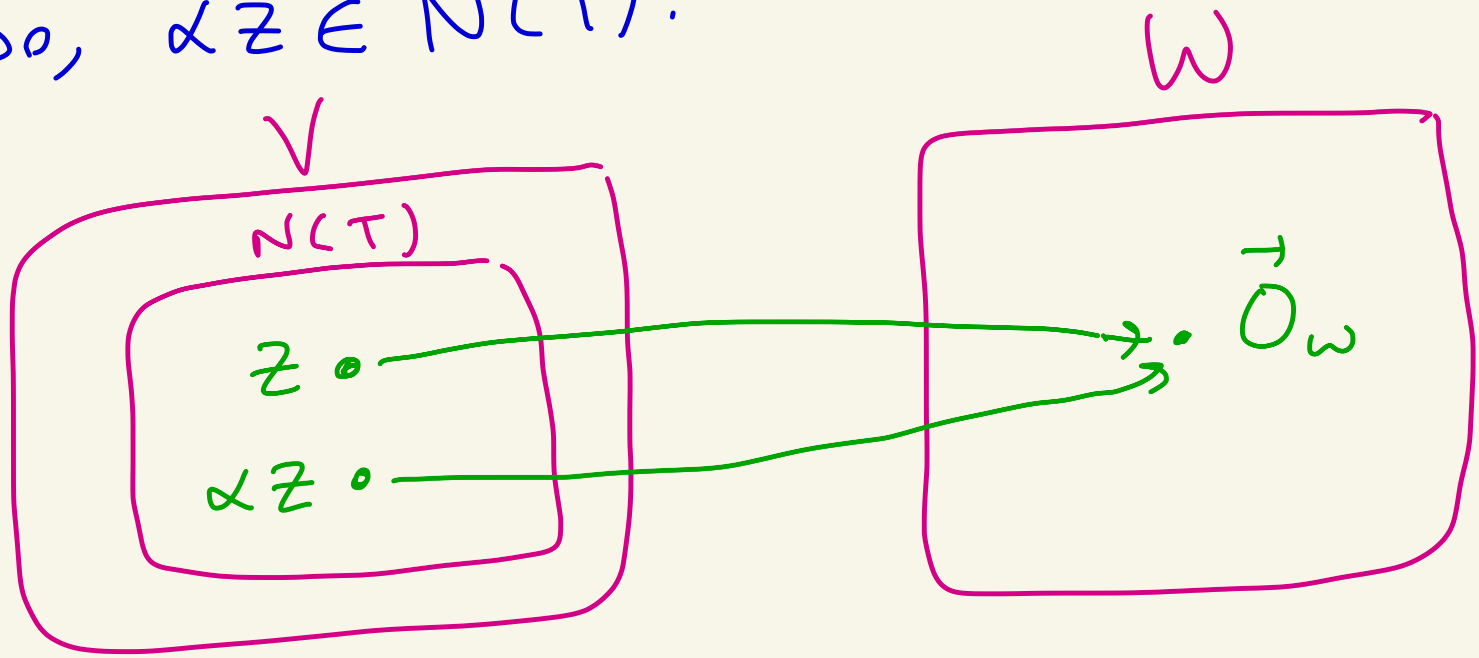
$$T(\alpha z) = \alpha T(z)$$

since T is linear

$$= \alpha \cdot \vec{0}_w = \vec{0}_w$$

Thus, $T(\alpha z) = \vec{0}_W$.

So, $\alpha z \in N(T)$.



By (i), (ii), and (iii) we have that $N(T)$ is a subspace of V.

2 Let's show $R(T)$ is a subspace of W.

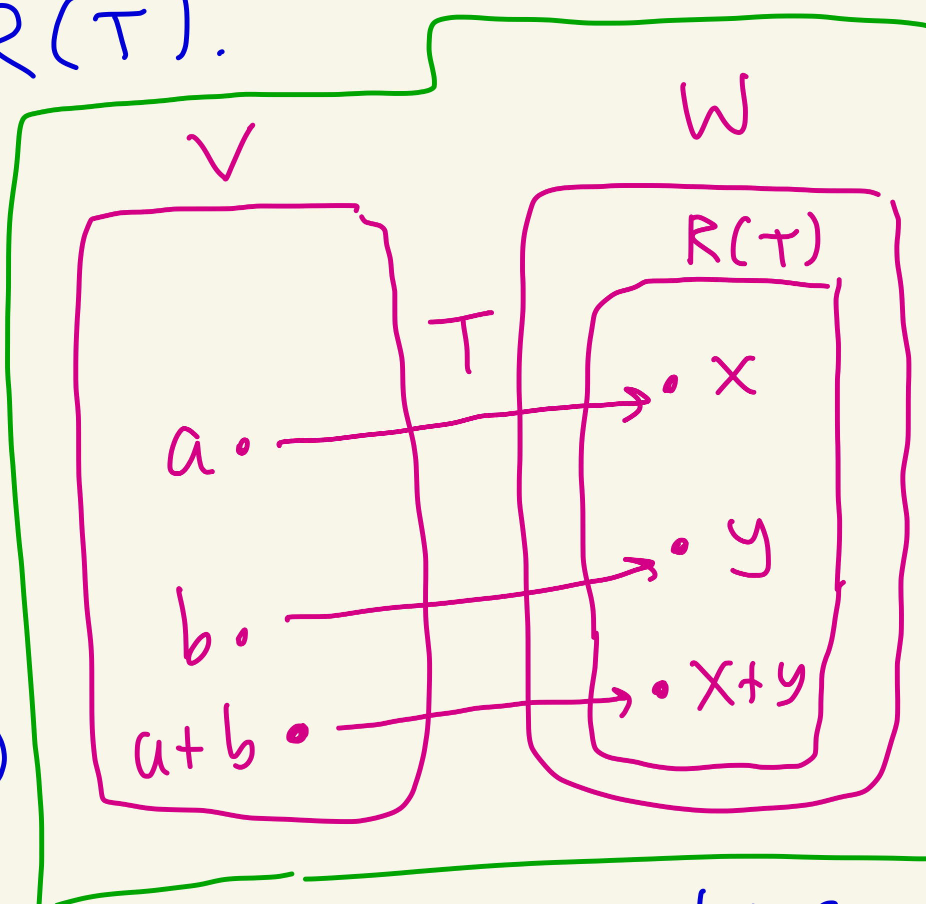
Recall $R(T) = \{ T(x) \mid x \in V \}$

(i) Because $\vec{0}_W = T(\vec{0}_V)$ and $\vec{0}_V \in V$ we know that $\vec{0}_W \in R(T)$.

(ii) Let's show $R(T)$ is closed under $+$.

Let $x, y \in R(T)$.

Then there exist $a, b \in V$ with $T(a) = x$ and $T(b) = y$.



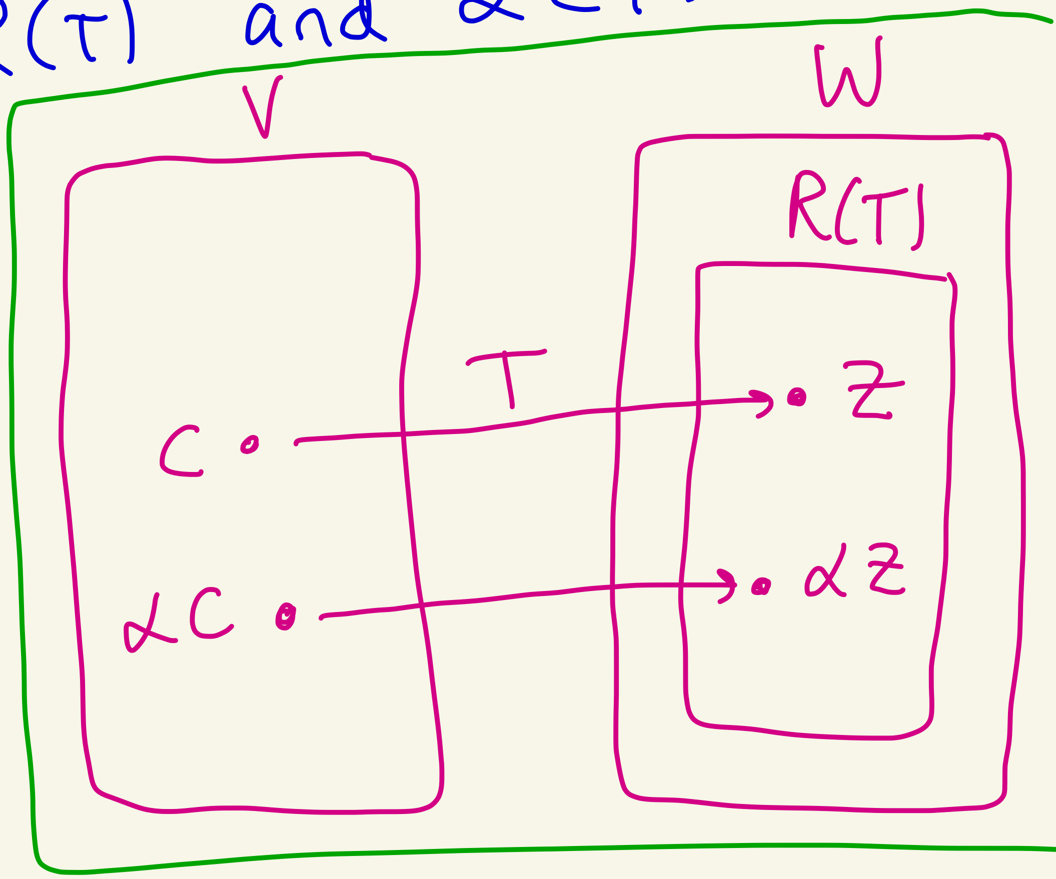
Thus,

$$x + y = T(a) + T(b) = T(a+b)$$

Since $x+y = T(a+b)$ and $a+b \in V$ we have that $x+y \in R(T)$.

(iii) Let's show $R(T)$ is closed under scalar mult.

Let $z \in R(T)$ and $\alpha \in F$.



Thus,

$z = T(c)$

where $c \in V$.

Then,

$\alpha z = \alpha T(c)$

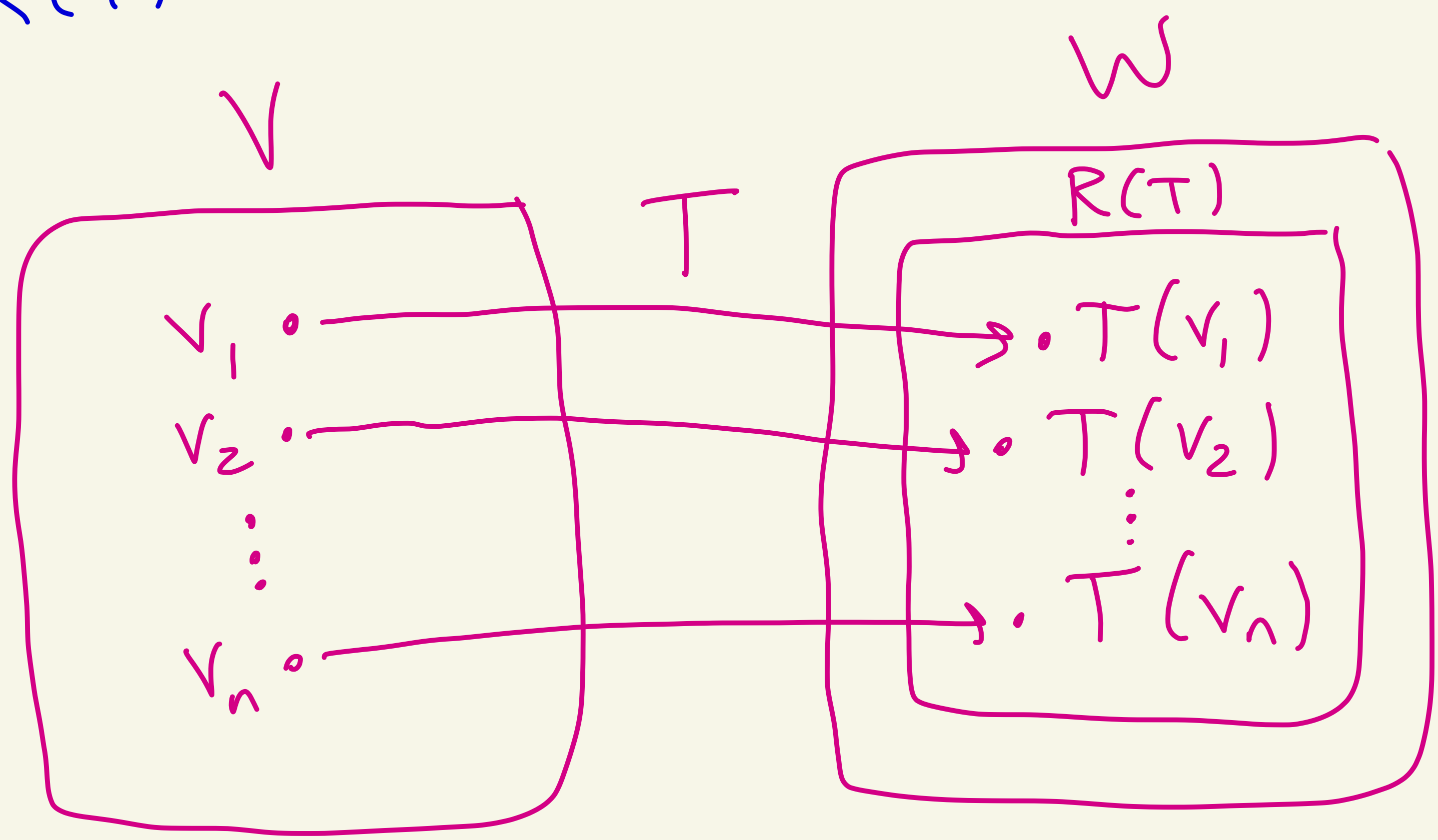
$= T(\alpha c)$.

Since $\alpha z = T(\alpha c)$ where $\alpha c \in V$ we know that $\alpha z \in R(T)$.

By (i), (ii), (iii), $R(T)$ is a subspace of W. $\blacksquare$

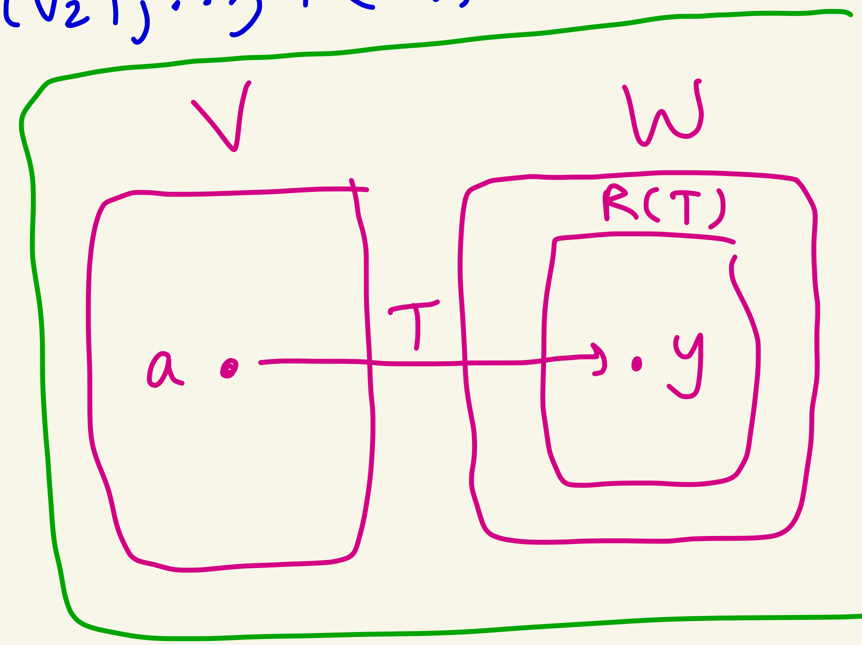
<u>Lemma:</u> Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

If $v_1, v_2, \ldots, v_n \in V$ and
$$V = \text{span}(\{v_1, v_2, \ldots, v_n\}),$$
then
$$R(T) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$$



proof: Suppose $v_1, v_2, \ldots, v_n \in V$ and $v_1, v_2, \ldots, v_n$ span V.

Lets show $T(v_1), T(v_2), \ldots, T(v_n)$ spans $R(T)$.



Let $y \in R(T)$. Then there exists $a \in V$ where $y = T(a)$.

Because $a \in V$ and $v_1, v_2, \ldots, v_n$ span V, we know that
$$a = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$
where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Thus,
$$y = T(a) = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$
$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$$

HW 3 problem since T is linear

So, $y \in \mathrm{Span}(\{T(v_1), \ldots, T(v_n)\})$.
So, $T(v_1), \ldots, T(v_n)$ span $R(T)$. $\blacksquare$

Rank-Nullity Theorem

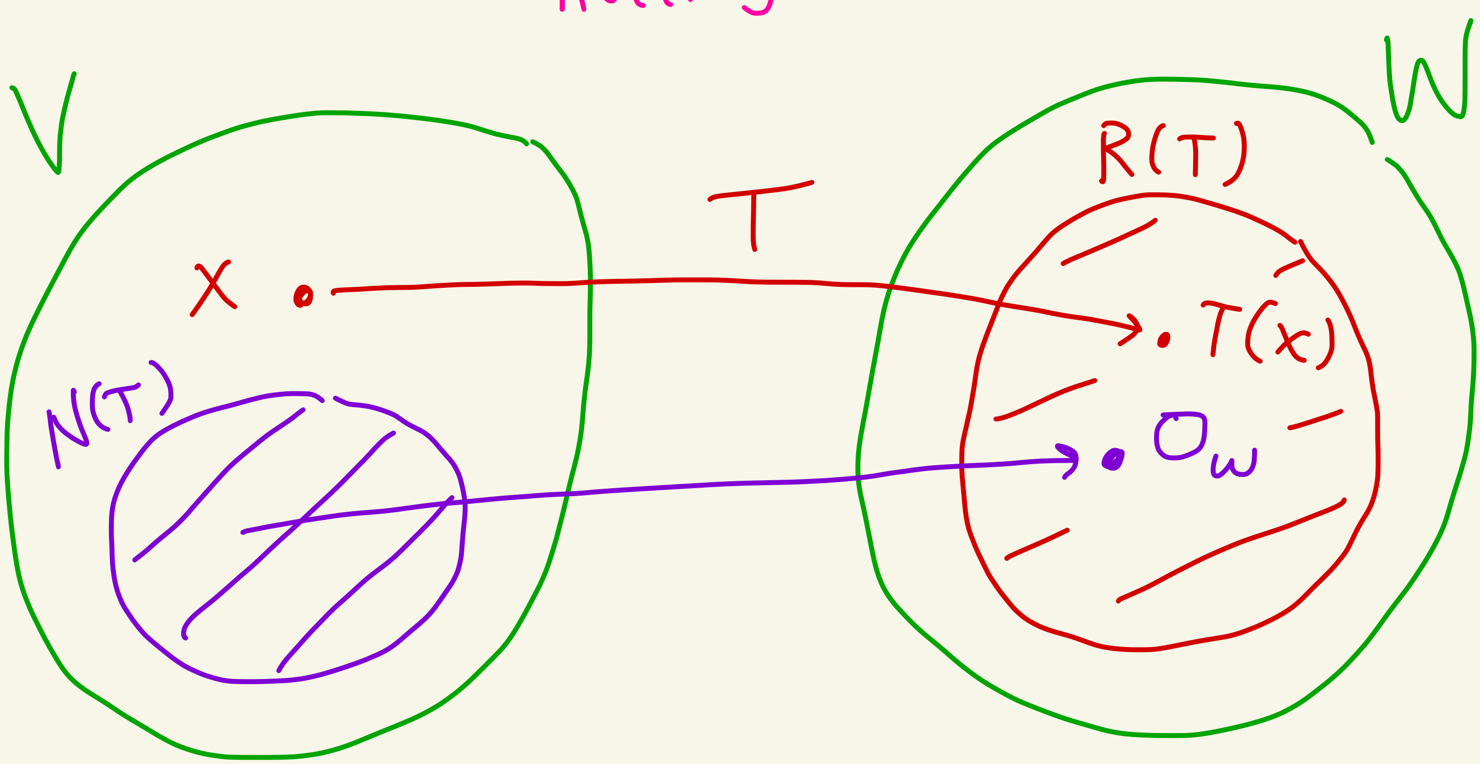
Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

If V is finite dimensional, then

① $N(T)$ is finite dimensional

② $R(T)$ is finite dimensional

and ③ $\dim(V) = \underbrace{\dim(N(T))}_{\text{nullity}(T)} + \underbrace{\dim(R(T))}_{\text{rank}(T)}$



<u>Proof:</u> Let $n = \dim(V)$.

By Monday's theorem, $N(T)$ is a subspace of V.

Thus, since V is finite dimensional, $N(T)$ is finite dimensional [Thm from class]

Also, if we set $k = \dim(N(T))$ then $k \leq n$. [Thm from class]

Thus, there exists a basis $\{v_1, v_2, \ldots, v_k\}$ for $N(T)$.

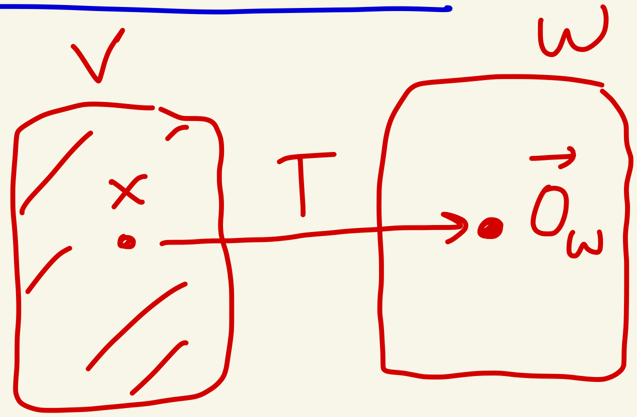
Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors for V and W.

Note that $T(\vec{0}_V) = \vec{0}_W$ and so $\vec{0}_W \in R(T)$.

Let's now break the proof into two cases.

Case 1: Suppose $R(T) = \{\vec{0}_W\}$

Then, $T(x) = \vec{0}_W$ for every $x \in V$.



Then, $N(T) = V$.

So, $\dim(R(T)) = 0$ and thus $R(T)$ is finite-dimensional

And,

$$\dim(V) = \dim(V) + 0$$

$$= \dim(N(T)) + \dim(R(T)).$$

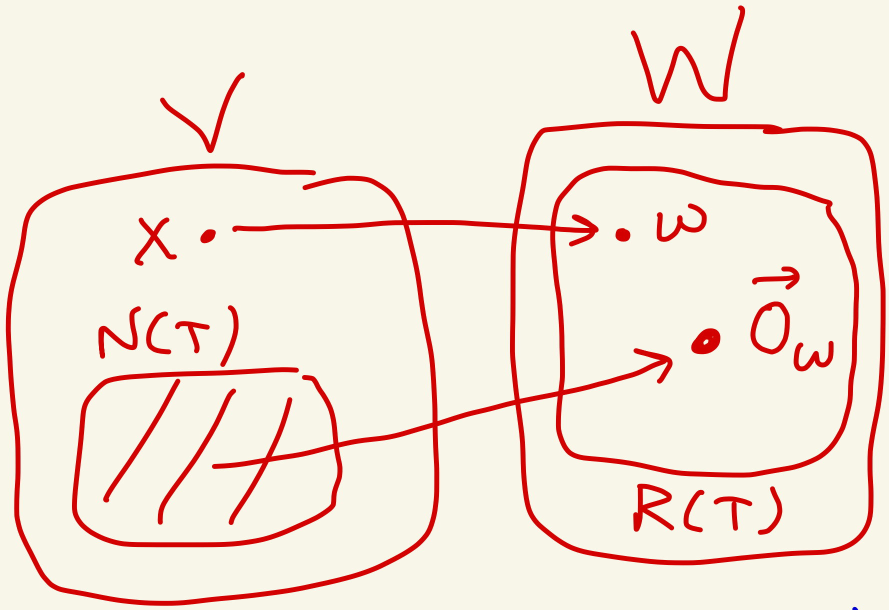
$V = N(T)$

$0 = \dim(R(T))$

Case 2: Suppose $R(T) \neq \{\vec{0}_W\}$

Then in this case $R(T)$ contains at least one non-zero vector $w \neq \vec{0}_W$

So, there exists $x \in V$ where $T(x) = w \neq \vec{0}_W$



Thus,

$N(T) \neq V.$

By HW 2 #9 we can extend the basis for $N(T)$ to all of V.

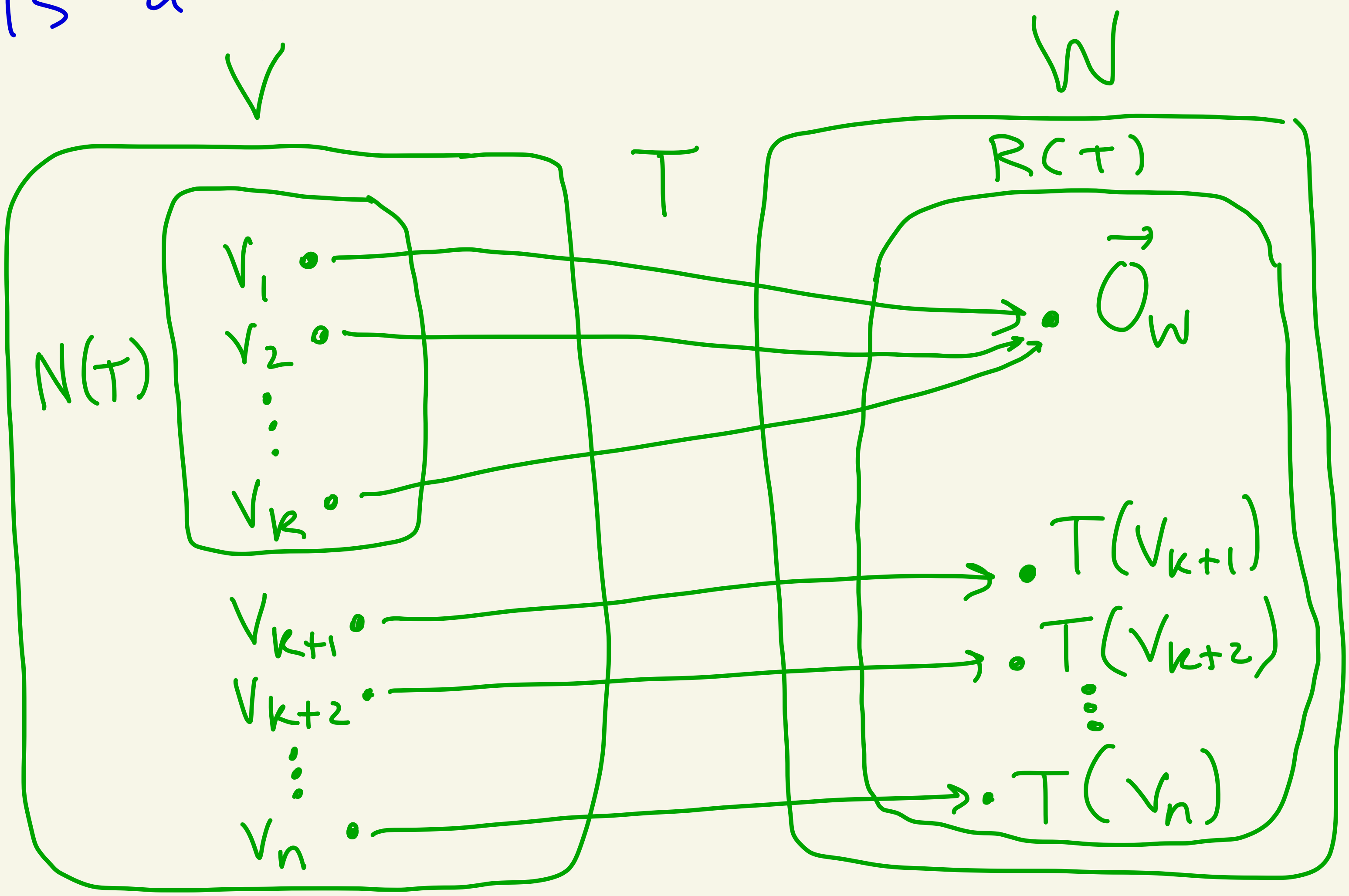
That is there exist

$$v_{k+1}, v_{k+2}, \dots, v_n \in \underbrace{V - N(T)}_{\text{in } V \text{ but not in } N(T)}$$

where

$$\beta = \{\underbrace{v_1, v_2, \dots, v_k}_{\text{basis for } N(T)}, \underbrace{v_{k+1}, v_{k+2}, \dots, v_n}_{\text{not in } N(T)}\}$$

is a basis for V.



We will show that

$$\beta' = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$$

is a basis for $R(T)$. (33)

Note once we've done this, then we will have finished the proof of the theorem because then $R(T)$ will be finite dimensional and

$$\begin{aligned} \dim(V) &= n \\ &= k + (n-k) \\ &= \dim(N(T)) + \left(\begin{matrix}\#\text{ elements} \\ \text{in } \beta'\end{matrix}\right) \\ &= \dim(N(T)) + \dim(R(T)). \end{aligned}$$

So, let's now show that β' is a basis for $R(T)$.

By a theorem from Monday, since $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ spans V, we know that

$$\begin{aligned} R(T) &= \operatorname{span}\big(\{T(v_1), T(v_2), \ldots, T(v_k), \\ &\qquad T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}\big) \\ &= \operatorname{span}\big(\{\vec{0}_W, \vec{0}_W, \ldots, \vec{0}_W, \\ &\qquad T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}\big) \\ &= \operatorname{span}\big(\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}\big) \end{aligned}$$

Thus, β' spans $R(T)$.

Let's now show β' is a linearly independent set.

Suppose

$$c_{k+1}T(v_{k+1}) + c_{k+2}T(v_{k+2}) + \dots + c_n T(v_n) = \vec{0}_W$$

Since T is linear we have

$$T(c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \dots + c_n v_n) = \vec{0}_W$$

Thus, $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \dots + c_n v_n$ is in $N(T)$.

Since $N(T)$ has $\{v_1, v_2, \dots, v_k\}$ as a basis we must have that

$$c_{k+1}v_{k+1} + \dots + c_n v_n = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

for some $c_1, c_2, \dots, c_k \in F$.

Thus,

$$-c_1 v_1 - \cdots - c_k v_k + c_{k+1} v_{k+1} + \cdots + c_n v_n = \vec{0}_V$$

But $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ is a basis for V and hence is linearly independent.

So the above equation implies that

$$-c_1 = -c_2 = \cdots = -c_k = c_{k+1} = \cdots = c_n = 0$$

In particular,

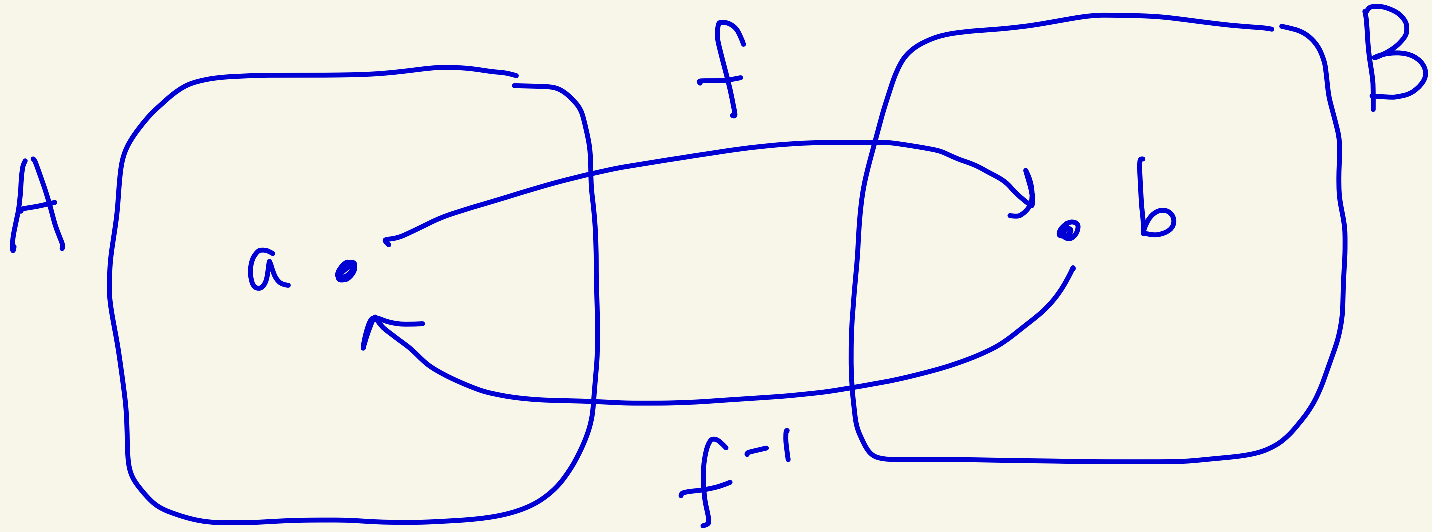
$$c_{k+1} = c_{k+2} = \cdots = c_n = 0.$$

Thus, $\beta' = \{T(v_{k+1}), \ldots, T(v_n)\}$ is linearly independent.

So, β' is a basis for $R(T)$. $\blacksquare$

<u>Recall:</u> Suppose $f: A \to B$ is 1-1 and onto where A and B are sets. Then $f^{-1}: B \to A$ is defined by

$f^{-1}(b) = a$ iff $f(a) = b$.



Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

Do in class: (Like HW problem 2)

- Show T is linear.
- Show $N(T) = \{\vec{0}\}$ and $\dim(N(T)) = 0 \longrightarrow T$ is 1-1
- Use rank-nullity to show $R(T) = \mathbb{R}^2 \longrightarrow T$ is onto

Let's find $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$.

$$T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

iff $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

iff $\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

iff $\boxed{\begin{matrix} a+b = c \\ a-b = d \end{matrix}}$

Let's solve this system.

$$\left(\begin{array}{cc|c} 1 & 1 & c \\ 1 & -1 & d \end{array}\right)$$

$$\xrightarrow{-R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 1 & c \\ 0 & -2 & d-c \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_2\to R_2} \left(\begin{array}{cc|c} 1 & 1 & c \\ 0 & 1 & -\frac{d}{2}+\frac{c}{2} \end{array}\right)$$

$\underbrace{\qquad\qquad}_{\text{row echelon form}}$

Thus,
$$\boxed{\begin{aligned} a+b&=c \quad &(1)\\ b&=-\frac{d}{2}+\frac{c}{2} \quad &(2)\end{aligned}}$$

(2) gives $b=-\frac{d}{2}+\frac{c}{2}$.

(1) gives $a=c-b=c-\left(-\frac{d}{2}+\frac{c}{2}\right)$
$$=\frac{1}{2}c+\frac{1}{2}d$$

Thus,

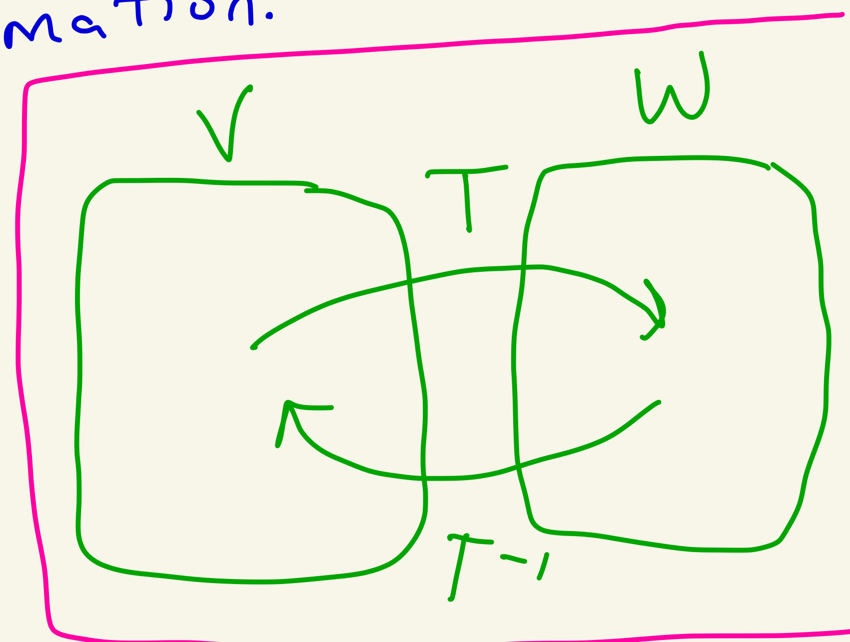
$$T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c + \frac{1}{2}d \\ \frac{1}{2}c - \frac{1}{2}d \end{pmatrix}$$

You can check that T^{-1} is linear by checking that

$$T^{-1}(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T^{-1}(v_1) + \alpha_2 T^{-1}(v_2)$$

for all $v_1, v_2 \in \mathbb{R}^2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a 1-1 and onto linear transformation. Then, $T^{-1}: W \to V$ is also a linear transformation.

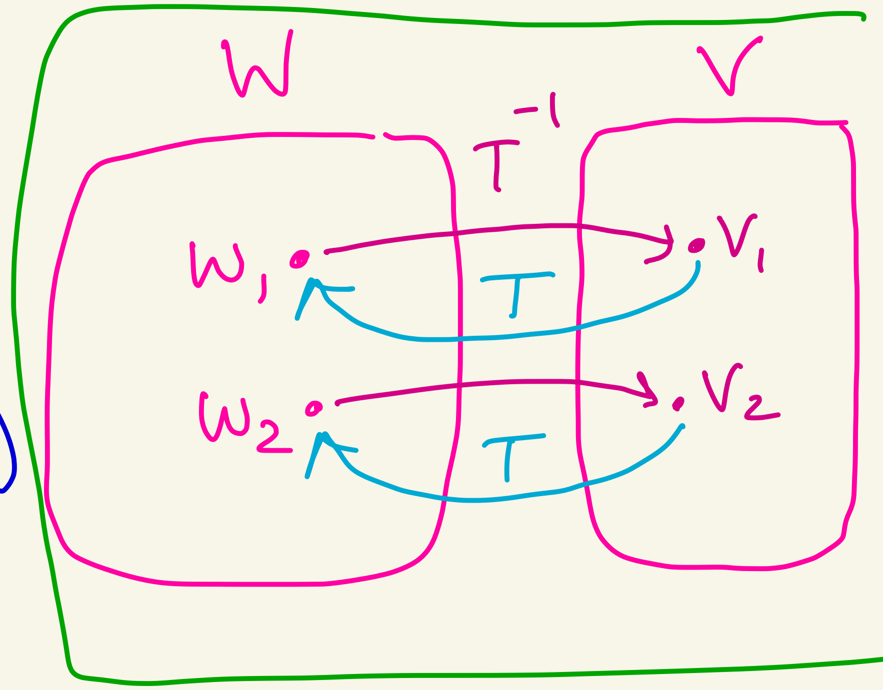


proof:

Because T is 1-1 and onto $T^{-1}: W \to V$ exists as a function. [MATH 3450]

We just need to show that T^{-1} is a linear transformation.

Let $\alpha_1, \alpha_2 \in F$ and $w_1, w_2 \in W$.



We will show that

$$T^{-1}(\alpha_1 w_1 + \alpha_2 w_2)$$
$$= \alpha_1 T^{-1}(w_1) + \alpha_2 T^{-1}(w_2)$$

Then, there exist $v_1, v_2 \in V$ where $T^{-1}(w_1) = v_1$ and $T^{-1}(w_2) = v_2$.

By def of inverse, $T(v_1) = w_1$ and $T(v_2) = w_2$.

Thus,

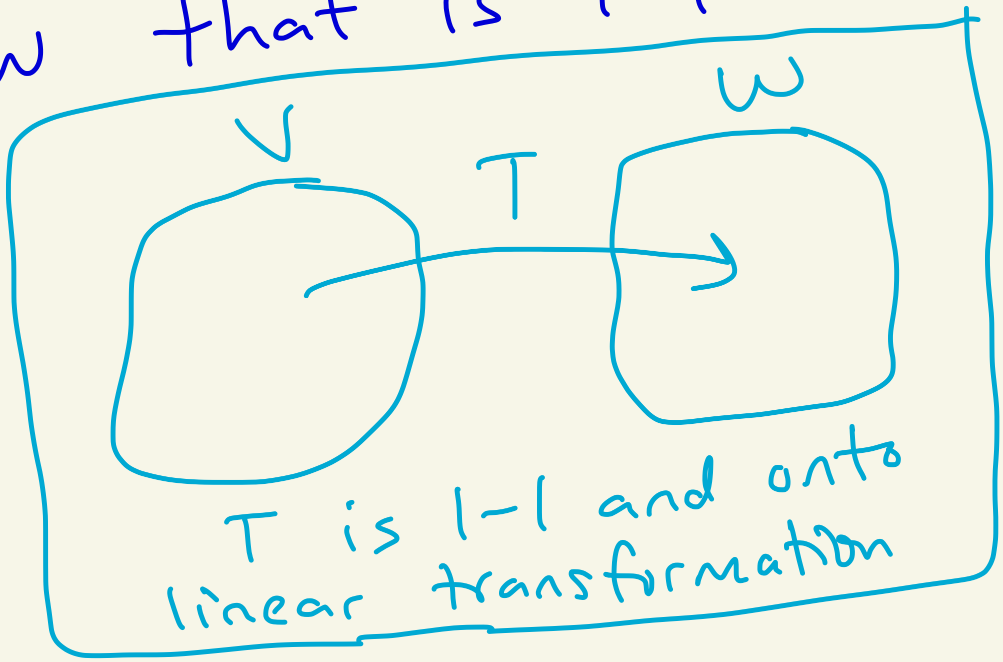
$$T^{-1}(\alpha_1 w_1 + \alpha_2 w_2) = T^{-1}(\alpha_1 T(v_1) + \alpha_2 T(v_2))$$
$$= T^{-1}(T(\alpha_1 v_1 + \alpha_2 v_2))$$
$$= \alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 T^{-1}(w_1) + \alpha_2 T^{-1}(w_2)$$

since T is linear

$T^{-1}(T(x)) = x$ for all $x \in V$, prop. of inverse

Def: Let V and W be vector spaces over a field F.

① An <u>isomorphism between V and W</u> is a linear transformation $T : V \to W$ that is 1-1 and onto.



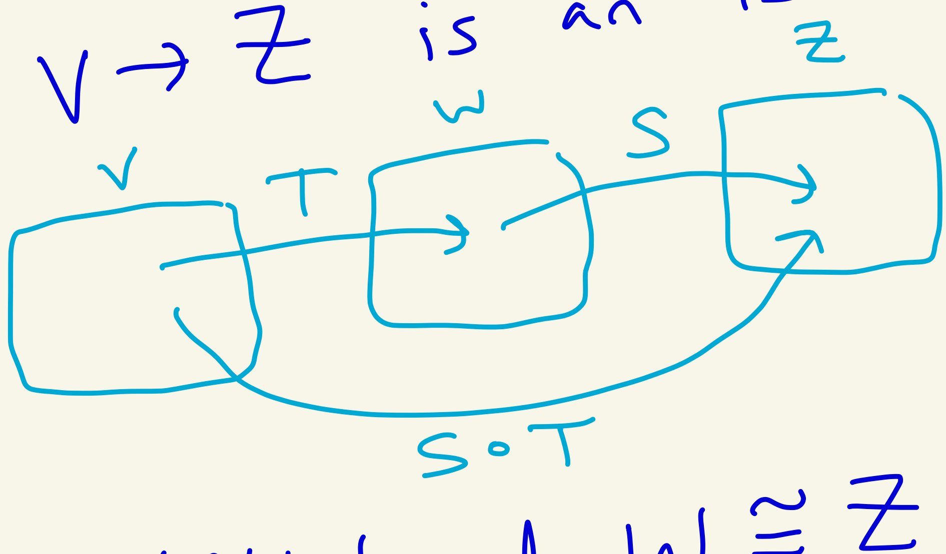
② We say that V and W are <u>isomorphic</u>, and write $V \cong W$, if there exists an isomorphism $T : V \to W$ between them.

<u>Note:</u> This def is well-defined by the following facts that one could show:

① If $T: V \to W$ is an isomorphism then $T^{-1}: W \to V$ is also an isomorphism.

Thus if $V \cong W$ then $W \cong V$.

② If $T: V \to W$ and $S: W \to Z$ are both isomorphisms, then $S \circ T: V \to Z$ is an isomorphism



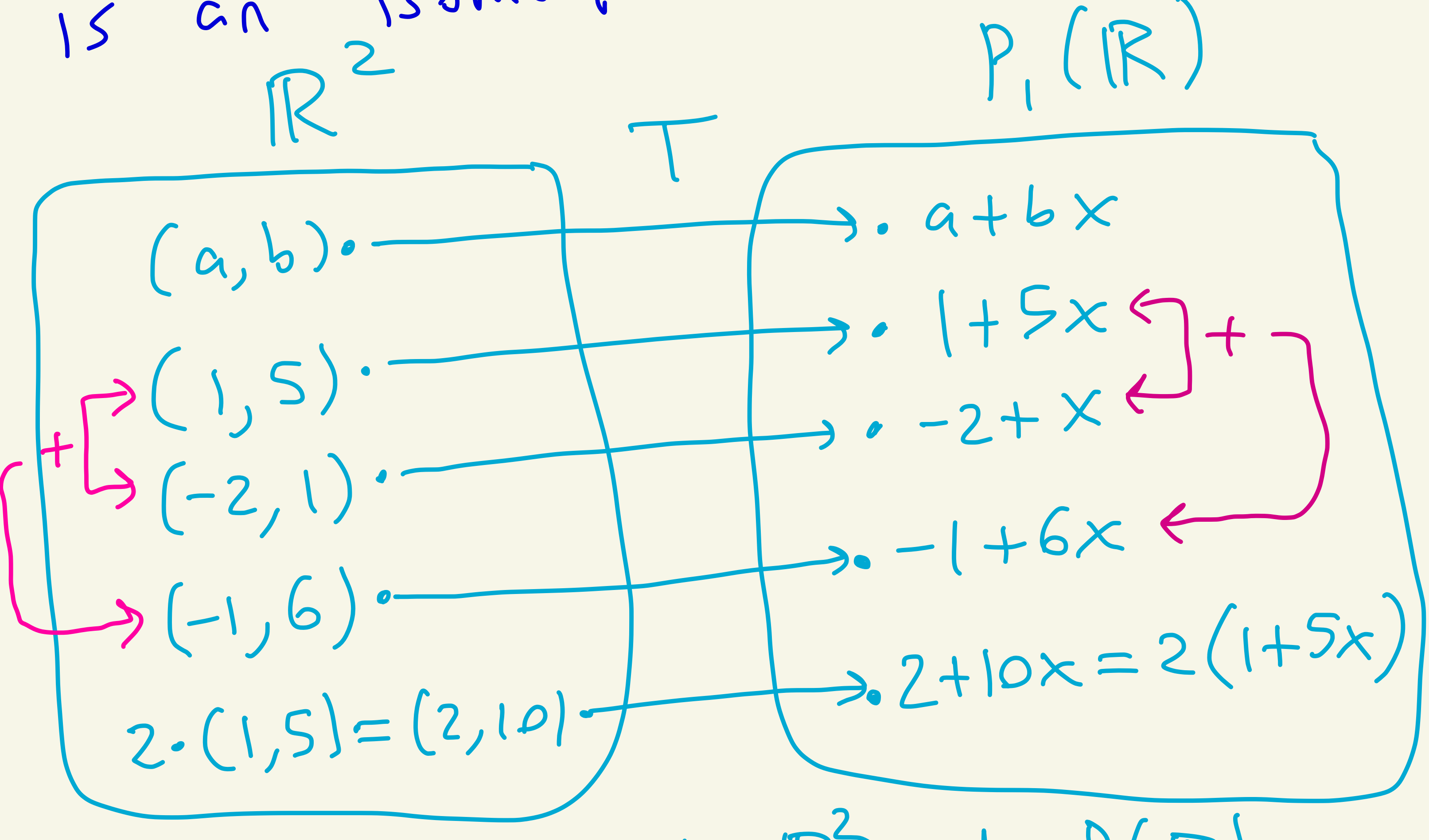
Thus if $V \cong W$ and $W \cong Z$ then $V \cong Z$.

$(S \circ T)(x) = S(T(x))$

Ex: Let $F = \mathbb{R}$. Let $V = \mathbb{R}^2$ and $W = P_1(\mathbb{R}) = \{a + bx \mid a, b \in \mathbb{R}\}$

Let $T: \mathbb{R}^2 \to P_1(\mathbb{R})$ be defined by $T((a,b)) = a + bx$

We will show later that T is an isomorphism.



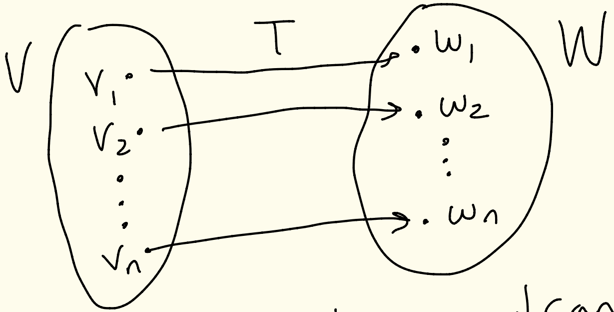
T is showing that $\mathbb{R}^2$ and $P_1(\mathbb{R})$ are structurally the same. The elements are just notated differently.

 [redacted]

Theorem: Let V and W be vector spaces over a field F. Suppose that V is finite-dimensional and $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

$\boxed{\text{part 1}}$ Let $w_1, w_2, \ldots, w_n \in W$.

① There exists a unique linear transformation $T: V \to W$ where $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$



this unique linear transformation is given by the formula

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \qquad (*)$$

② T given above is an isomorphism iff $\beta' = \{w_1, w_2, \ldots, w_n\}$ is a basis for W.

 [redacted]

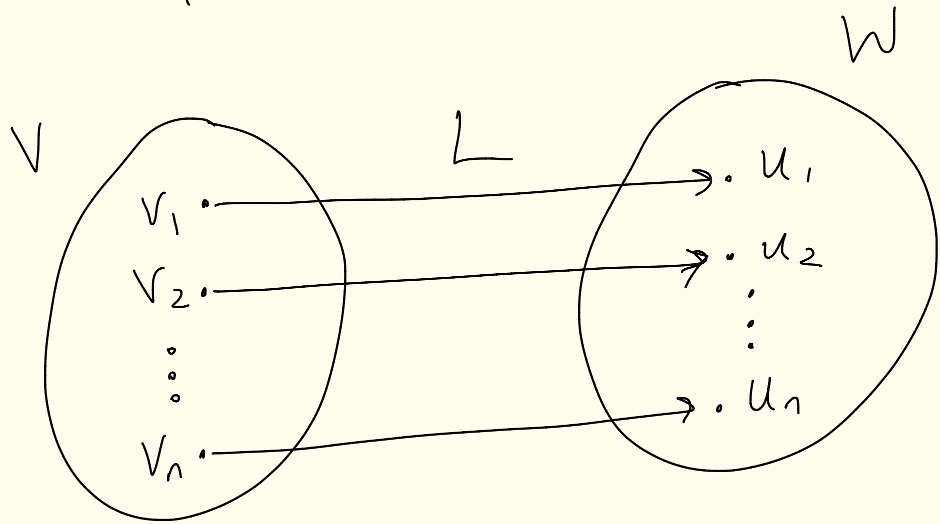
part 2

All linear transformations between V and W are constructed as in 1 above. That is, if $L: V \to W$ is a linear transformation, set

$$u_i = L(v_i) \quad \text{for } i = 1, 2, \ldots, n$$

and then the formula for L is

$$L(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$
$$= c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$



[redacted]

proof: $\boxed{\text{part 1}}$

1 Let T be defined by $(*)$.

That is,

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$$

for any $c_i \in F$.

Let's show T is a linear transformation and $T(v_i) = w_i$ for all i.

Why is T linear?

Let $x, y \in V$ and $\alpha, \delta \in F$.

Since β is a basis for V, we can write $x = e_1 v_1 + \dots + e_n v_n$ and $y = d_1 v_1 + \dots + d_n v_n$ where $e_i, d_i \in F$. Then,

$$T(\alpha x + \delta y)$$
$$= T\big(\alpha(e_1 v_1 + \dots + e_n v_n) + \delta(d_1 v_1 + \dots + d_n v_n)\big)$$
$$= T\big((\alpha e_1 + \delta d_1) v_1 + \dots + (\alpha e_n + \delta d_n) v_n\big) =$$

[redacted]

$$= T\big((\alpha e_1 + \delta d_1)v_1 + \cdots + (\alpha e_n + \delta d_n)v_n\big)$$

$$\overset{(*)}{=} (\alpha e_1 + \delta d_1)w_1 + \cdots + (\alpha e_n + \delta d_n)w_n$$

$$= \alpha e_1 w_1 + \cdots + \alpha e_n w_n$$
$$+ \delta d_1 w_1 + \cdots + \delta d_n w_n$$

$$= \alpha(e_1 w_1 + \cdots + e_n w_n)$$
$$+ \delta(d_1 w_1 + \cdots + d_n w_n)$$

$$\overset{(*)}{=} \alpha T(e_1 v_1 + \cdots + e_n v_n)$$
$$+ \delta T(d_1 v_1 + \cdots + d_n v_n)$$

$$= \alpha T(x) + \delta T(y).$$

So, T is linear.

Also,

$$T(v_1) = T(1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n) = 1 \cdot w_1 = w_1$$
$$\vdots$$
$$T(v_n) = T(0 \cdot v_1 + 0 \cdot v_2 + \cdots + 1 \cdot v_n) = 1 \cdot w_n = w_n$$

So, $T(v_i) = w_i$ for all i.

 [redacted]

Why is T unique?

Suppose $S: V \to W$ is another linear transformation with $S(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

Let $x \in V$.

Then, since β is a basis for V,

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

And,

$$\begin{aligned} S(x) &= S(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\ &\overset{\text{S is linear}}{=} c_1 S(v_1) + c_2 S(v_2) + \cdots + c_n S(v_n) \\ &\overset{S(v_i) = w_i}{=} c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \\ &= T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\ &\overset{\text{def of } T}{=} T(x) \end{aligned}$$

So, $S = T$ on V.

So, T is the unique linear transf. with $T(v_i) = w_i \ \forall i$

[redacted]

2) T defined by $(*)$ is an isomorphism iff $\beta' = \{w_1, w_2, \dots, w_n\}$ is a basis for W.

($\Leftarrow$) Suppose β' is a basis for W. Let's show that T defined by $(*)$ is 1-1 and onto, and hence an isomorphism

$\boxed{1\text{-}1}$: Suppose $T(x) = T(y)$ for some $x, y \in V$.

Since β is a basis for V, $x = c_1 v_1 + \dots + c_n v_n$ and $y = d_1 v_1 + \dots + d_n v_n$ for $c_i, d_i \in F$.

Since $T(x) = T(y)$, by def of T, we have

$$\underbrace{c_1 w_1 + \dots + c_n w_n}_{T(x)} = \underbrace{d_1 w_1 + \dots + d_n w_n}_{T(y)}$$

So, $(c_1 - d_1) w_1 + \dots + (c_n - d_n) w_n = \vec{0}$

By assumption, β' is a lin. ind. set, so

$$0 = c_1 - d_1 = c_2 - d_2 = \dots = c_n - d_n$$

 [redacted]

So, $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$
and hence

$$x = c_1 v_1 + \cdots + c_n v_n = d_1 v_1 + \cdots + d_n v_n = y.$$

$\boxed{\text{onto}}$: We need to show $R(T) = W$.
By a previous thm, since $\beta = \{v_1, v_2, \ldots, v_n\}$ spans V, we know $R(T) = \text{span}(\{T(v_1), \ldots, T(v_n)\})$.

So,

$$R(T) = \text{span}(\{T(v_1), \ldots, T(v_n)\})$$
$$= \text{span}(\{w_1, \ldots, w_n\})$$
$$= W.$$

we are assuming $\beta' = \{w_1, \ldots, w_n\}$ is a basis for W

So, T is onto W.

Thus, T is an isomorphism.

[redacted]

($\Rightarrow$) Now suppose T is an isomorphism, ie 1-1 and onto.
Let's show β' is a basis for W.

Since T is onto, $R(T) = W$.

Therefore,
$$W = R(T) = \operatorname{span}(\{T(v_1), \ldots, T(v_n)\})$$
$$= \operatorname{span}(\{w_1, \ldots, w_n\})$$

So, β' spans W.
Is β' a lin. ind. set ?

Suppose
$$d_1 w_1 + \ldots + d_n w_n = \vec{0}_W$$
where $d_i \in F$.
Since T is 1-1 and onto, T^{-1} exists and is linear (from Monday) and $T^{-1}(w_i) = v_i$ for $i = 1, \ldots, n$.

Since T^{-1} is linear, $T^{-1}(\vec{0}_w) = \vec{0}_v$. [redacted]

So,

$$\vec{0}_v = T^{-1}(\vec{0}_w) = T^{-1}(d_1 w_1 + \dots + d_n w_n)$$

$$= d_1 T^{-1}(w_1) + \dots + d_n T^{-1}(w_n)$$

$$= d_1 v_1 + \dots + d_n v_n$$

Since $\beta = \{v_1, \dots, v_n\}$ is a basis and $\vec{0}_v = d_1 v_1 + \dots + d_n v_n$

we get $d_1 = d_2 = \dots = d_n = 0$.

Thus, β' is a lin. ind. set.

since if $d_1 w_1 + \dots + d_n w_n = \vec{0}_w$

then $d_1 = d_2 = \dots = d_n = 0$.

So, β' is a basis for W.

[redacted]

part 2

55

Suppose L is a linear transformation and $u_i = L(v_i)$ for $i = 1, 2, \dots, n$.

Then,

$$L(c_1 v_1 + \dots + c_n v_n)$$

$$= c_1 L(v_1) + \dots + c_n L(v_n)$$

$$= c_1 u_1 + \dots + c_n u_n.$$

L is linear

$\blacksquare$

Ex: Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let's make a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$.

Step 1: Pick a basis for $V = \mathbb{R}^3$.

Let's use the standard basis

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2, v_3\}$$

(from theorem)

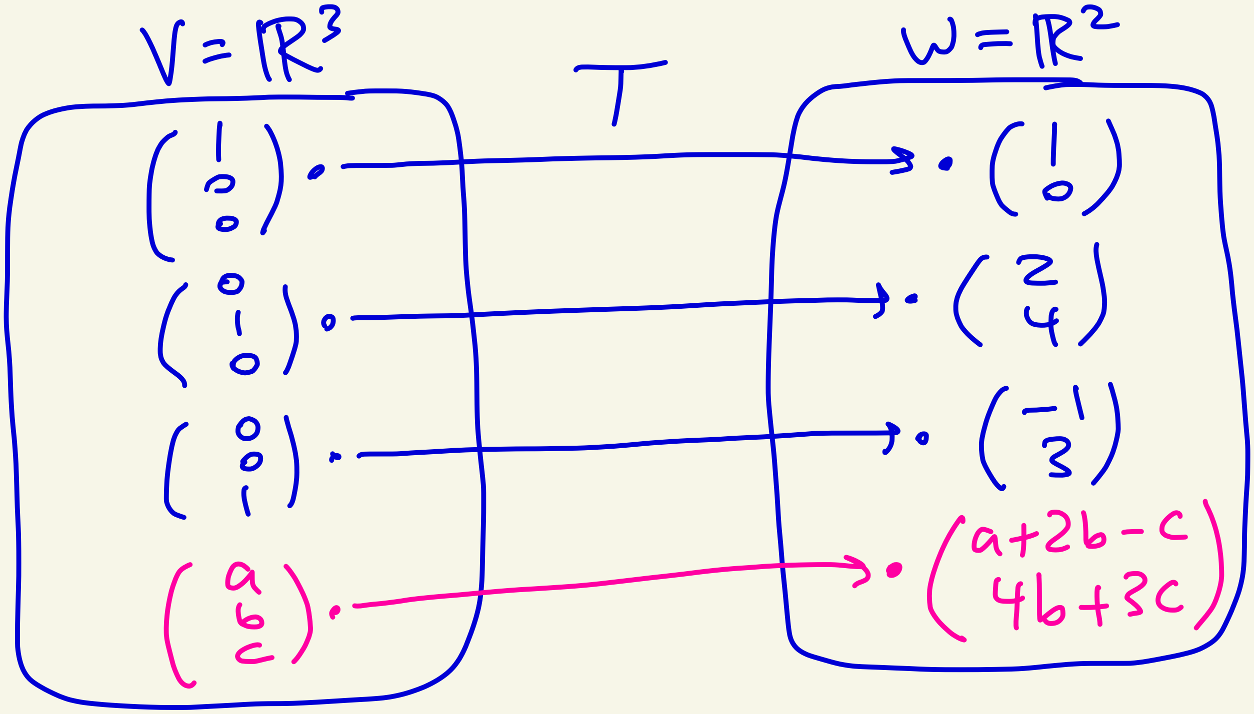
Step 2: Decide where β goes.

Pick:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_1$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = w_2$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 3 \end{pmatrix}}_{\text{can put any vectors here}} = w_3$$



Then in general for any $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ we have from the theorem that

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T\left(a \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$= a \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{w_1} + b \cdot \underbrace{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}_{w_2} + c \cdot \underbrace{\begin{pmatrix} -1 \\ 3 \end{pmatrix}}_{w_3}$$

$$= \begin{pmatrix} a+2b-c \\ 4b+3c \end{pmatrix}$$

T will be a linear transformation.

Ex: $V = \mathbb{R}^2$

$W = P_1(\mathbb{R}) = \{a + bx \mid a, b \in \mathbb{R}\}$

Let's build a linear transformation between these vector spaces.

Step 1: Pick a basis for $V = \mathbb{R}^2$.

Let's pick the standard basis $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

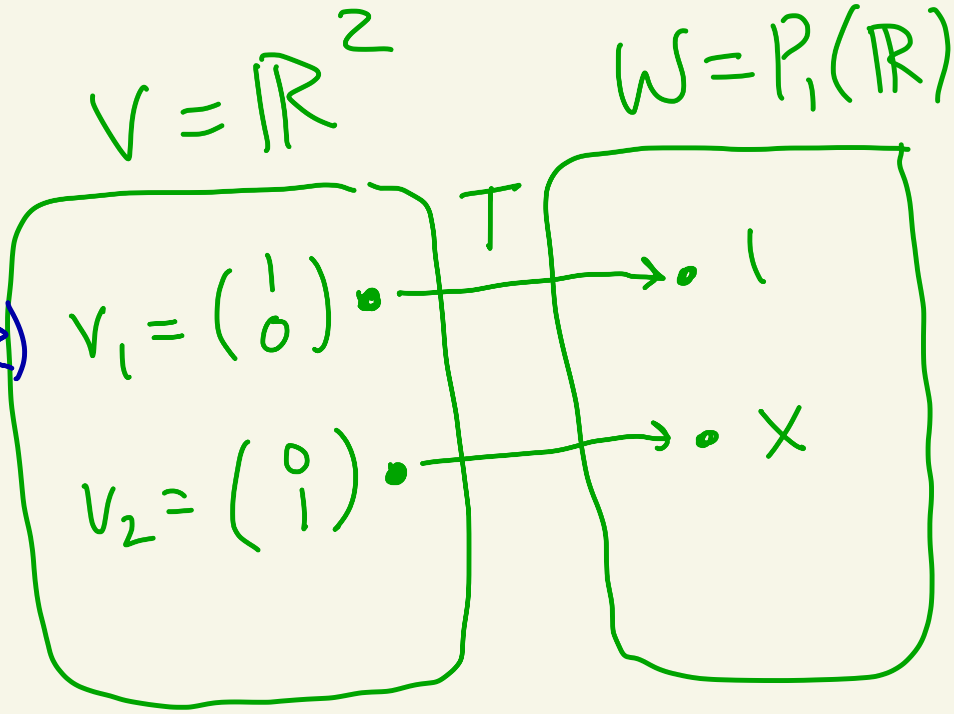
Step 2: Choose where each element of β goes.

You can send them anywhere.

Define $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ where

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x$



There is only one way to make this linear transformation.

And this is described in Mondays Theorem. The reason is as follows:

Suppose we have $v \in \mathbb{R}^2$.

Then $v = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a, b \in \mathbb{R}$.

So, to define T on v we need

$$T(v) = T\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= T\left(a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\underline{\underline{\ast}}\; aT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

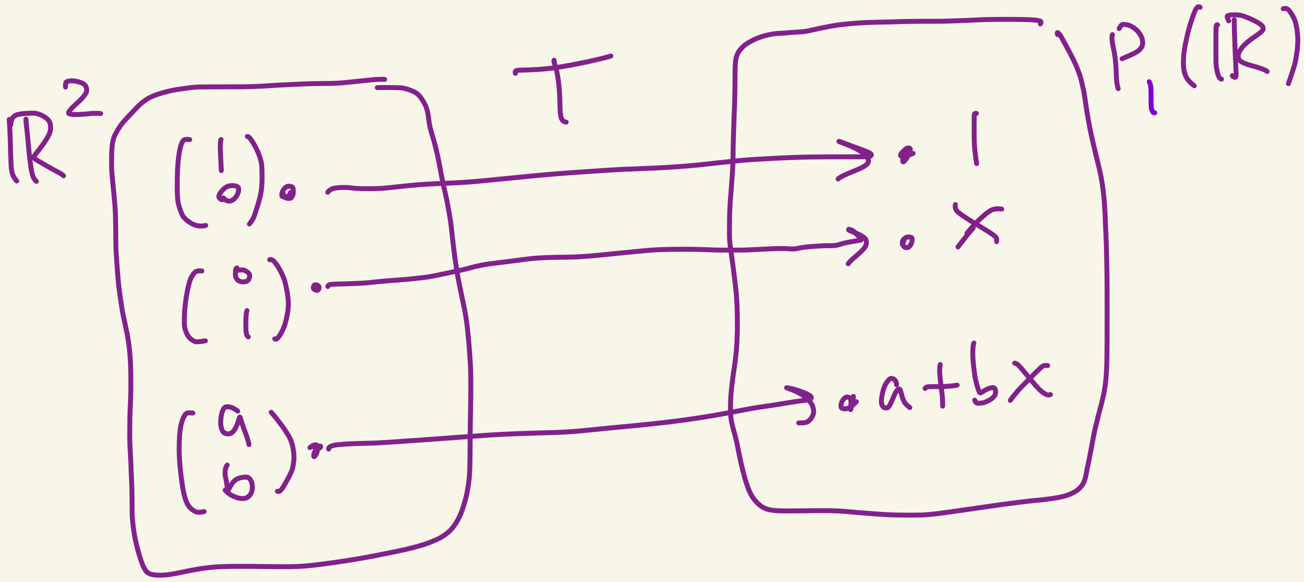
In order for T to be linear we need

$$= a \cdot 1 + b \cdot x$$

$$= a + bx$$

This is what the theorem said also.

Thus the only linear transformation $T: \mathbb{R}^2 \to P_1(\mathbb{R})$ where $T\begin{pmatrix}1\\0\end{pmatrix} = 1$ and $T\begin{pmatrix}0\\1\end{pmatrix} = x$ is given by the formula $T\begin{pmatrix}a\\b\end{pmatrix} = a + bx$



By Mondays theorem this is a linear transformation.

Furthermore, it is an isomorphism if and only if $\{1, x\}$ is a basis for $P_1(\mathbb{R})$ which it is!

Thus, T is an isomorphism and
$$\mathbb{R}^2 \cong P_1(\mathbb{R}).$$

Ex: Let's consider the vector spaces $V=\mathbb{R}^4$ and $W=M_{2,2}(\mathbb{R})$.

Pick the standard basis for $V=\mathbb{R}^4$ which is $\beta=\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$.

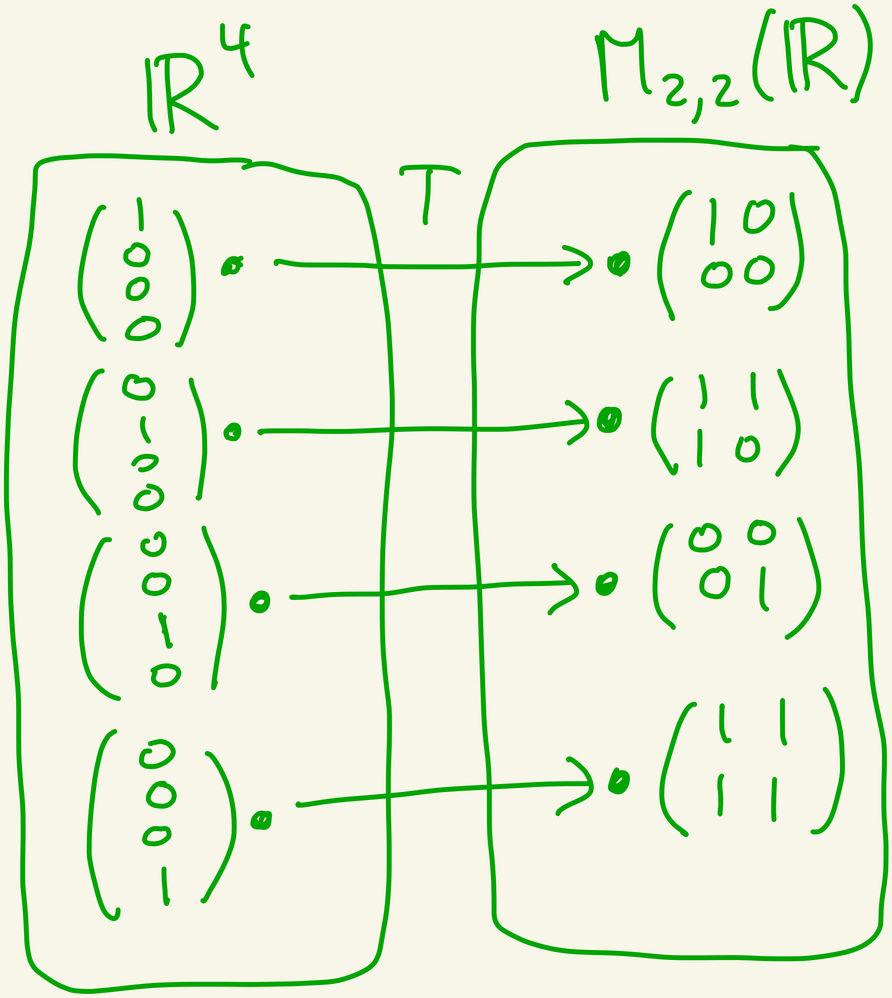
Let's create the linear transformation where

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}1&1\\1&0\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$$



The formula for such a linear transformation is given by

$$T\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = T\left(a\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+d\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$$

$$= aT\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+bT\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+cT\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+dT\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

has to be true to make T linear as in theorem from Mon.

$$= a\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\begin{pmatrix}1&1\\1&0\end{pmatrix}+c\begin{pmatrix}0&0\\0&1\end{pmatrix}+d\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

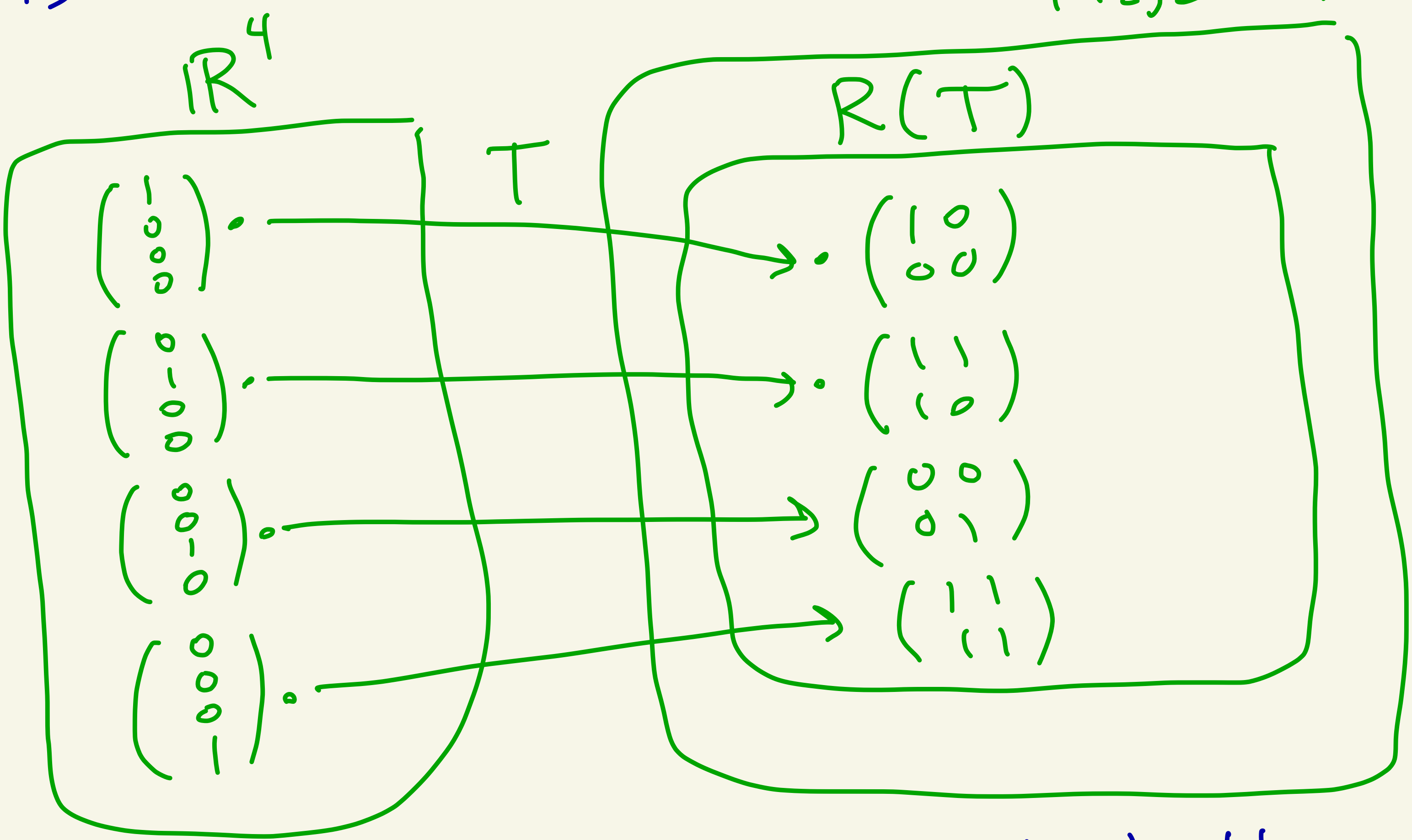
$$= \begin{pmatrix}a+b+d & b+d\\ b+d & c+d\end{pmatrix}$$

Thus from Mon theorem $T: \mathbb{R}^4 \to M_{2,2}(\mathbb{R})$ given by

$$T\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}a+b+d & b+d\\ b+d & c+d\end{pmatrix}$$

is a linear transformation.

The theorem from Monday tells us that T is an isomorphism iff $\beta' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis for $M_{2,2}(\mathbb{R})$.



Idea is: If β' spans $M_{2,2}(\mathbb{R})$, then $R(T) = \operatorname{span}(\beta') = M_{2,2}(\mathbb{R})$ and T will be onto. If in addition, β' is a lin. ind. set that will make T one-to-one.

β' is actually <u>not</u> lin. ind. because a solution to the equation

$$c_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+c_2\begin{pmatrix}1&1\\1&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\0&1\end{pmatrix}+c_4\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

is

$$0\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+1\cdot\begin{pmatrix}1&1\\1&0\end{pmatrix}+1\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}-1\cdot\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

which shows β' is a lin. dep. set.

So, T will not be an isomorphism.

If you wanted to, you could show that $\dim(N(T))=1$ by solving

$$T\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a+b+d & b+d\\ b+d & c+d\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

By rank-nullity,

$$\underbrace{\dim(\mathbb{R}^4)}_{4}=\underbrace{\dim(N(T))}_{1}+\dim(R(T))$$

So, $\dim(R(T))=3$ and so T is not onto.

Theorem: Let V and W be finite-dimensional vector spaces over a field F.
We have that $V \cong W$ if and only if $\dim(V) = \dim(W)$.

proof:

($\Leftarrow$) Suppose $\dim(V) = \dim(W)$.
Then there exist bases
$\beta = \{v_1, v_2, \ldots, v_n\}$ for V and
$\beta' = \{w_1, w_2, \ldots, w_n\}$ for W
where $n = \dim(V) = \dim(W)$.

Construct the linear transformation $T: V \to W$ given as follows:

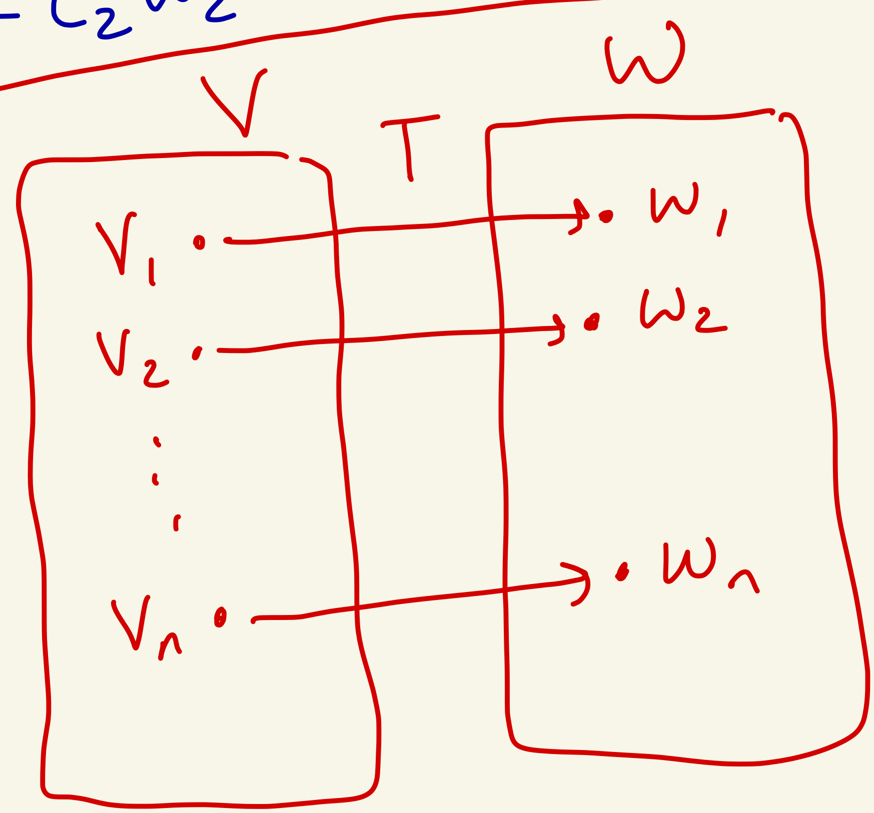
Given $x \in V$, express x in terms of the basis β as follows:

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

Then, as in Mondays thm, define

$$T(x) = T(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)$$
$$= c_1 w_1 + c_2 w_2 + \ldots + c_n w_n$$

So, β goes to β'. Since β' is a basis for W, by Mon. thm, T is an isomorphism.



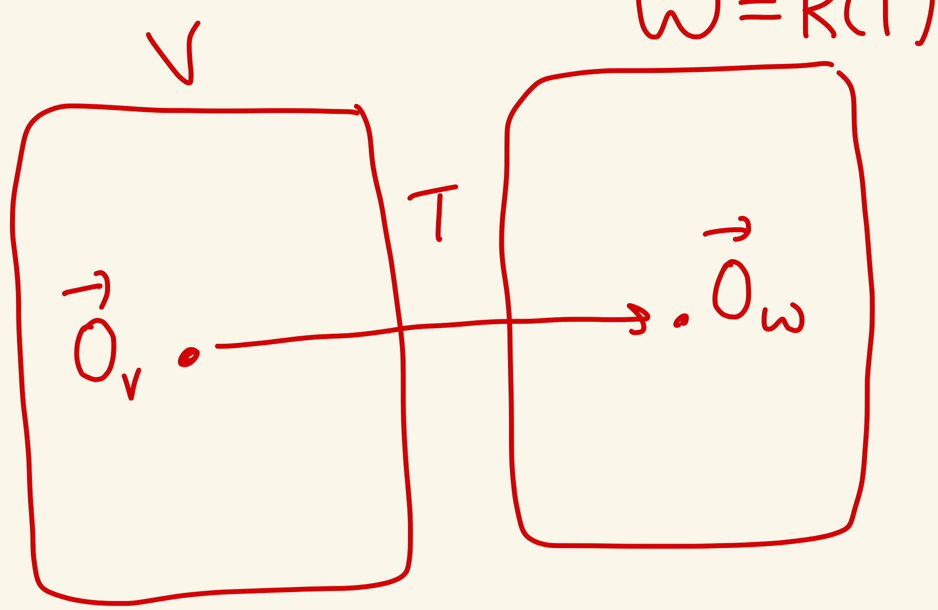
($\Rightarrow$) Suppose V and W are isomorphic.

This means there exists an isomorphism $T : V \longrightarrow W$.

So, T is a linear transformation that is 1-1 and onto.

By HW, because T is 1-1 we know that $N(T) = \{\vec{0}_V\}$.

Because T is onto we know $R(T) = W$.



By the rank-nullity theorem,

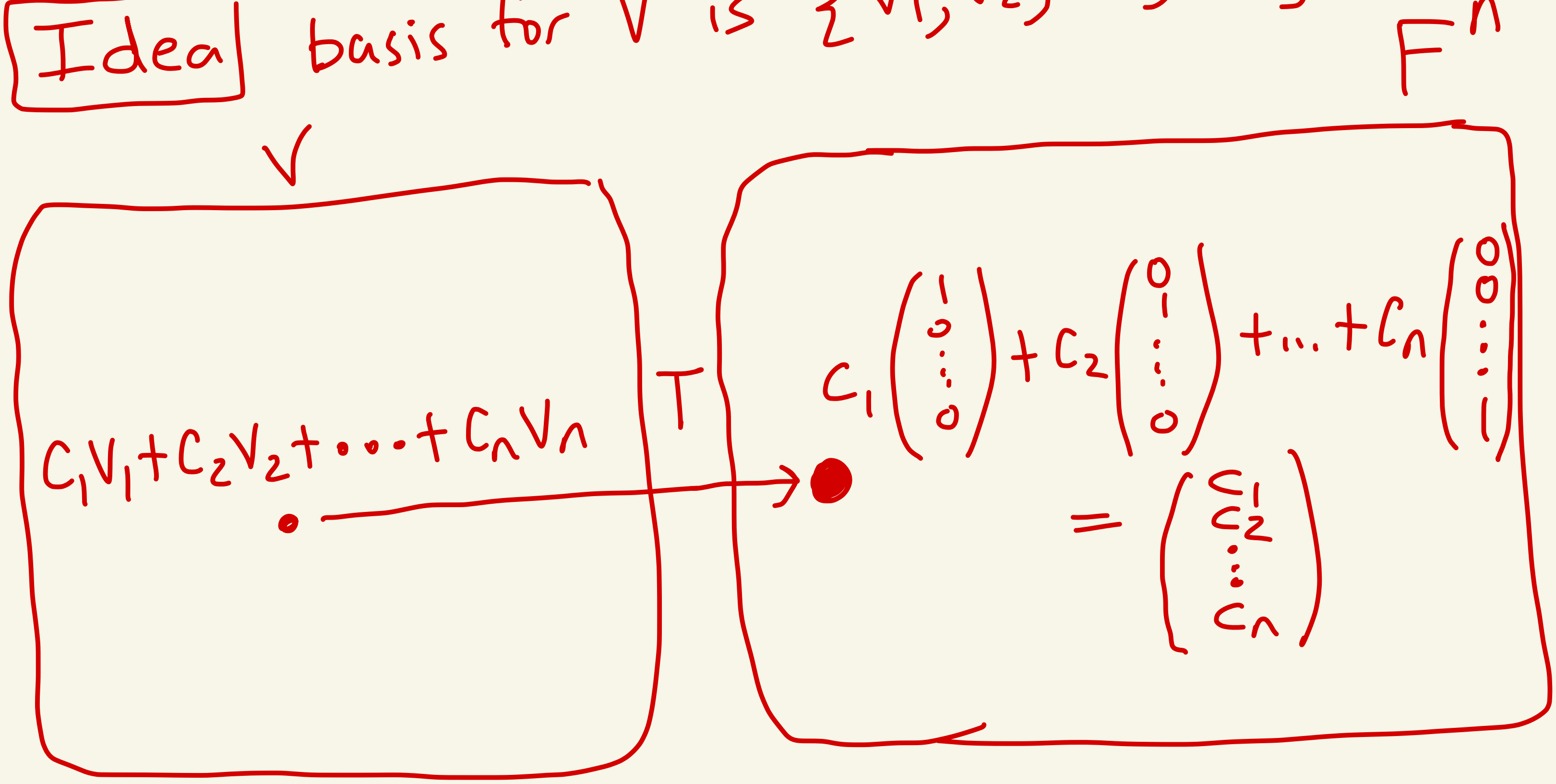
$$\begin{aligned} \dim(V) &= \dim(N(T)) + \dim(R(T)) \\ &= \dim(\{\vec{0}_V\}) + \dim(W) \\ &= 0 + \dim(W) = \dim(W). \end{aligned}$$

$\blacksquare$

<u>Corollary:</u> Let V be a finite-dimensional vector space over a field F. If $\dim(V) = n$, then $V \cong F^n$.

<u>proof:</u> Use the previous theorem and the fact that $\dim(F^n) = n = \dim(V)$. So, $V \cong F^n$. ▨

$\boxed{\text{Idea}}$ basis for V is $\{v_1, v_2, \ldots, v_n\}$



That is,

$$T(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

is an isomorphism between V and F^n.

Some people use the term "invertible" instead of "isomorphism"