Review topic -

Review of determinants

<u>Def:</u> Let A be an $n \times n$ matrix with coefficients from a field F. Let $1 \leq i \leq n$ and $1 \leq j \leq n$. The matrix A_{ij} is defined to be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and j-th column of A.

<u>Ex:</u> $A = \begin{pmatrix} 1 & 5 & 7 \\ 0 & -1 & 2 \\ 3 & \pi & 10 \end{pmatrix}$

$A_{32} = \begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix}$

$A_{13} = \begin{pmatrix} 0 & -1 \\ 3 & \pi \end{pmatrix}$

$\begin{pmatrix} 1 & \cancel{5} & 7 \\ 0 & \cancel{-1} & 2 \\ \cancel{3} & \cancel{\pi} & \cancel{10} \end{pmatrix}$

$\begin{pmatrix} \cancel{1} & \cancel{5} & \cancel{7} \\ 0 & -1 & \cancel{2} \\ 3 & \pi & \cancel{10} \end{pmatrix}$

<u>Def:</u> Let A be an $n \times n$ matrix with coefficients from a field F.
Let a_{ij} be the entry in the i-th row and j-th column of A.

① If $n=1$ and $A = (a_{11})$ then define $\det(A) = a_{11}$

② If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then define $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

③ If $n \geq 3$, then define $\det(A)$ as follows. Pick a column j where $1 \leq j \leq n$. Define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\substack{\text{sum over rows } i \\ \text{column } j \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(\underbrace{A_{ij}}_{\substack{\text{this is an} \\ (n-1)\times(n-1) \\ \text{matrix}}})$$

This is called <u>the expansion of the determinant along the j-th column</u>

<u>Note:</u> One can also expand along a row in part 3 of the previous definition. To do this, pick a row i with $1 \leq i \leq n$ and replace step 3 with

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\substack{\text{sum over the columns } j \\ \text{row } i \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(A_{ij})$$

This is called the <u>expansion of the determinant along row i.</u>

<u>Fact:</u> This def is well-defined. One can show that the final result is the same no matter what row or column you expand on in step 3.

<u>Notation:</u> One can also use bars instead of det. For example,

$$\det\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ \pi & 5 & 7 \end{pmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ \pi & 5 & 7 \end{vmatrix}$$

Ex: $\det(10) = 10$

Ex: $\det\begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} = (-1)(7) - (0)(3) = -7$

Ex: Let $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

Expand on row $i=1$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\det(A) = \underbrace{(-1)^{1+1} a_{11} \det(A_{11})}_{i=1,\ j=1} + \underbrace{(-1)^{1+2} a_{12} \det(A_{12})}_{i=1,\ j=2}$$

$$+ \underbrace{(-1)^{1+3} a_{13} \det(A_{13})}_{i=1,\ j=3}$$

$$= \underbrace{(1)(3)\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix}} + \underbrace{(-1)(1)\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix}} + \underbrace{(1)(0)\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}}$$

a_{11} $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$ $\qquad$ a_{12} $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$ $\qquad$ a_{13} $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

$$= (3)\left[(-4)(-2) - (3)(4)\right] + (-1)\left[(-2)(-2) - (3)(5)\right] + 0$$

$$= (3)(-4) - [-11] = -1$$

(5)

So,

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

Useful tool

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

put + in top left and alternate +/-

Ex: Let $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Lets expand on column 2.

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= \overset{-}{(-1)}(1)\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + \overset{+}{(1)}(-4)\begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix} + \overset{-}{(-1)}(4)\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (-1)\left[4-15\right] - 4\left[-6-0\right] - 4\left[9-0\right]$$

$$= (-1)(-11) + 24 - 36 = 35 - 36 = -1$$

Ex: Let $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}$

Let's expand on column $j=1$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

gives the $(-1)^{i+j}$

$$\det(A) = \underbrace{\overset{+}{(1)}(1)\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}}$$

$$\underbrace{\overset{-}{(-1)}(0)\begin{vmatrix} 2 & -1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}} + \underbrace{\overset{+}{(1)}(0)\begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -2 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}} + \underbrace{\overset{-}{(-1)}(0)\begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 0 \\ -2 & -4 & 3 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}}$$

$$= \underbrace{\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}}_{\text{calculated previously}} + 0 + 0 + 0$$

$$= -1$$

Properties of the determinant

Let F be a field and A and B be $n \times n$ matrices with entries from F. Then:

1. $\det(AB) = \det(A)\det(B)$
2. A is invertible iff $\det(A) \neq 0$

   If A is invertible then
   $\det(A^{-1}) = (\det(A))^{-1}$