

Review topic -

Review of determinants



①

Def: Let A be an $n \times n$ matrix with coefficients from a field F . Let $1 \leq i \leq n$ and $1 \leq j \leq n$. The matrix A_{ij} is defined to be the $(n-1) \times (n-1)$ matrix obtained by removing the i -th row and j -th column of A .

Ex: $A = \begin{pmatrix} 1 & 5 & 7 \\ 0 & -1 & 2 \\ 3 & \pi & 10 \end{pmatrix}$

$$A_{32} = \begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 0 & -1 \\ 3 & \pi \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & 7 \\ 0 & -1 & 2 \\ 3 & \pi & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 5 & 7 \\ 0 & -1 & 2 \\ 3 & \pi & 10 \end{pmatrix}$$

Def: Let A be an $n \times n$ matrix with coefficients from a field F .

Let a_{ij} be the entry in the i -th row and j -th column of A .

① If $n=1$ and $A = (a_{11})$ then define $\det(A) = a_{11}$

② If $n=2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then define $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

③ If $n \geq 3$, then define $\det(A)$ as follows. Pick a column j where $1 \leq j \leq n$. Define

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

sum over rows i
column j is fixed
this is an
($n-1$) \times ($n-1$)
matrix

This is called the expansion of the determinant along the j -th column

Note: One can also expand along a row in part ③ of the previous definition. To do this, pick a row i with $1 \leq i \leq n$ and replace step ③ with

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

sum over the columns j
row i is fixed

This is called the expansion of the determinant along row i .

Fact: This def is well-defined.

One can show that the final result is the same no matter what row or column you expand on in step 3.

Notation: One can also use bars instead of \det . For example,

$$\det \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ \pi & 5 & 7 \end{pmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ \pi & 5 & 7 \end{vmatrix}$$

Ex: $\det(10) = 10$

Ex: $\det \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} = (-1)(7) - (0)(3) = -7$

Ex: Let $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

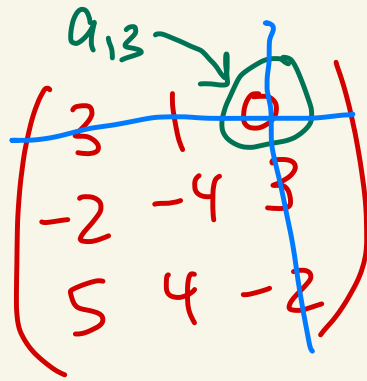
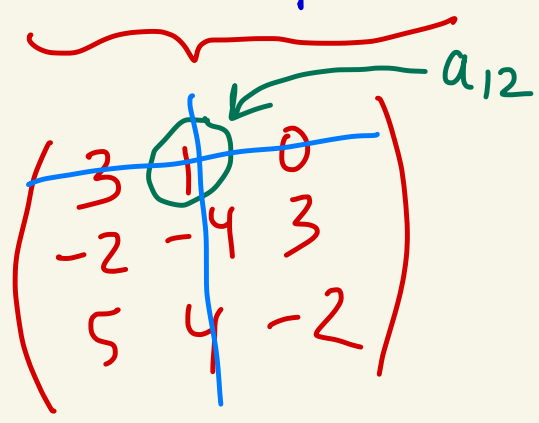
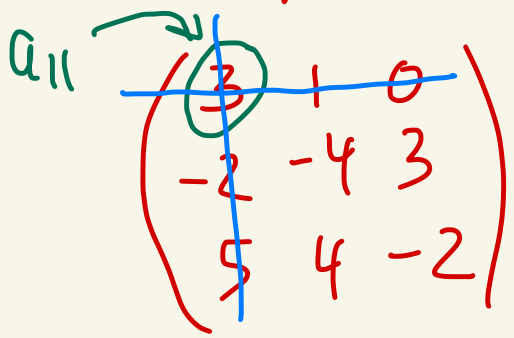
Expand on row $i=1$

$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

$\det(A) = \underbrace{(-1)^{1+1} a_{11} \det(A_{11})}_{i=1, j=1} + \underbrace{(-1)^{1+2} a_{12} \det(A_{12})}_{i=1, j=2}$

$+ \underbrace{(-1)^{1+3} a_{13} \det(A_{13})}_{i=1, j=3}$

$= (1)(3) \begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} + (-1)(1) \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + (1)(0) \begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$



$$= (3) [(-4)(-2) - (3)(4)] + (-1) [(-2)(-2) - (3)(5)] + 0$$

$$= (3)(-4) - [-11] = \textcircled{-1}$$

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So,

$$\det \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

Useful tool

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

← put + in top left and alternate +/-

Ex: Let $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

(6)

Lets expand on column 2.

$$\det \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= \underbrace{(-1)(1) \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix}}_{\text{Term 1}} + \underbrace{(1)(-4) \begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix}}_{\text{Term 2}} + \underbrace{(-1)(4) \begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix}}_{\text{Term 3}}$$

$$\begin{pmatrix} \cancel{3} & 1 & \cancel{0} \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & \cancel{1} & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ \cancel{5} & \cancel{4} & \cancel{-2} \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (-1)[4-15] - 4[-6-0] - 4[9-0]$$

$$= (-1)(-11) + 24 - 36 = 35 - 36 = -1$$

Ex: Let $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}$

Let's expand on column $j=1$

$$\det(A) = \underbrace{(1)(1)}_{+} \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$$

+	-	+	-
-	+	-	+
+	-	+	-
-	+	-	+

gives the $(-1)^{i+j}$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}$$

$$\underbrace{(-1)(0)}_{-} \begin{vmatrix} 2 & -1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} + \underbrace{(1)(0)}_{+} \begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -2 \end{vmatrix} + \underbrace{(-1)(0)}_{-} \begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 0 \\ -2 & -4 & 3 \end{vmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} + 0 + 0 + 0$$

calculated
previously

$$= -1$$

Properties of the determinant

Let F be a field and A and B be $n \times n$ matrices with entries from F . Then :

- ① $\det(AB) = \det(A) \det(B)$
- ② A is invertible iff $\det(A) \neq 0$
If A is invertible then
 $\det(A^{-1}) = (\det(A))^{-1}$