Topic 4 –
The matrix of a
linear transformation

HW 4 Topic
The Matrix of a linear Transformation

Def: Let V be a finite-dimensional vector space over a field F. Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. We write $\beta = [v_1, v_2, \ldots, v_n]$ to mean that β is an <u>ordered basis</u> for V, that is, the order of the vectors in β is given and fixed.

Def: Let V be a vector space over a field F with an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$.

Let $x \in V$.

Then we can write x uniquely in the form

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

We write

$$[x]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

and call $[x]_\beta$ the <u>coordinates of x with respect to β</u>

(or the <u>coordinate vector of x with respect to β</u>)

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Consider $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

You can check that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are linearly independent.

Since are two linearly independent vectors and $\dim(V) = \dim(\mathbb{R}^2) = 2$ we know that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are a basis.

Thus, $\beta = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$ is an ordered basis.

Pick $x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Let's find $[x]_\beta$

We need to solve

$$\underbrace{\begin{pmatrix} 5 \\ 4 \end{pmatrix}}_{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

coordinates for x

This becomes

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ c_2 \end{pmatrix}$$

which becomes

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ 2c_1 + c_2 \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} 5 &= c_1 - c_2 \\ 4 &= 2c_1 + c_2 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 3 & -6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} c_1 - c_2 &= 5 \\ c_2 &= -2 \end{aligned}}$$

Thus, $c_2 = -2$.

And, $c_1 = 5 + c_2 = 5 - 2 = 3$.

So,

$$x = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus,

$$[x]_\beta = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

What if we kept $x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ but changed to the standard basis $\beta' = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$.

Then, $x = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $[x]_{\beta'} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Ex: Let

$V = P_2(\mathbb{R}) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$

$F = \mathbb{R}$

Let

$\beta = [1,\ 1+x,\ 1+x+x^2]$

You can show that these 3 vectors are lin. ind. Since $\dim(P_2(\mathbb{R})) = 3$ they must be a basis

Consider

$v = 2 - x + 3x^2$

Let's find $[v]_\beta$.

We need to solve

$$\underbrace{2 - x + 3x^2}_{v} = c_1 \cdot 1 + c_2(1+x) + c_3(1+x+x^2)$$

v's coordinates with respect to β

This becomes

$$2 - x + 3x^2 = (c_1 + c_2 + c_3) \cdot 1 + (c_2 + c_3) \cdot x + c_3 x^2$$

So we get

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ c_2 + c_3 &= -1 \\ c_3 &= 3 \end{aligned}}$$

This is already a reduced system

We get

$$c_3 = 3$$

$$c_2 = -1 - c_3 = -1 - 3 = -4$$

$$c_1 = 2 - c_2 - c_3 = 2 - (-4) - 3 = 3$$

Thus,

$$2 - x + 3x^2 = 3 \cdot 1 - 4 \cdot (1 + x) + 3 \cdot (1 + x + x^2)$$

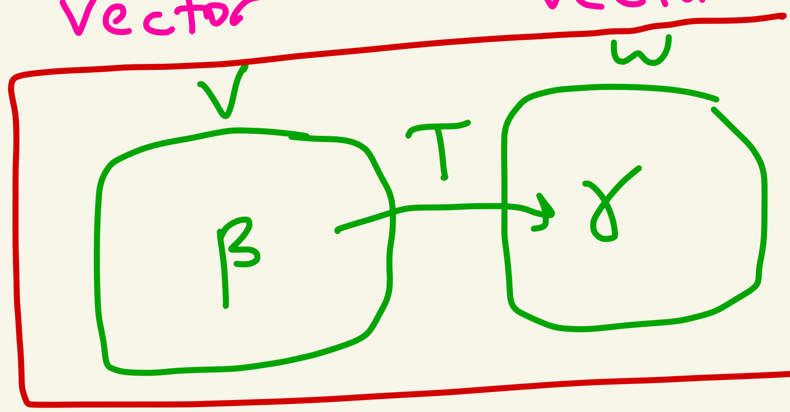
And $\left[2 - x + 3x^2\right]_\beta = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$

<u>**Def:**</u> Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V and let γ be an ordered basis for W.

The matrix

$$[T]_\beta^\gamma = \left(\underbrace{[T(v_1)]_\gamma}_{\text{column vector}} \;\Bigg|\; \underbrace{[T(v_2)]_\gamma}_{\text{column vector}} \;\Bigg|\; \cdots \;\Bigg|\; \underbrace{[T(v_n)]_\gamma}_{\text{column vector}} \right)$$

is called the <u>matrix of T with respect to β and γ.</u>



If $V = W$ and $\beta = \gamma$, then we just write $[T]_\beta$ instead of $[T]_\beta^\beta$

Ex: Let $V = W = \mathbb{R}^2$ and $F = \mathbb{R}$.

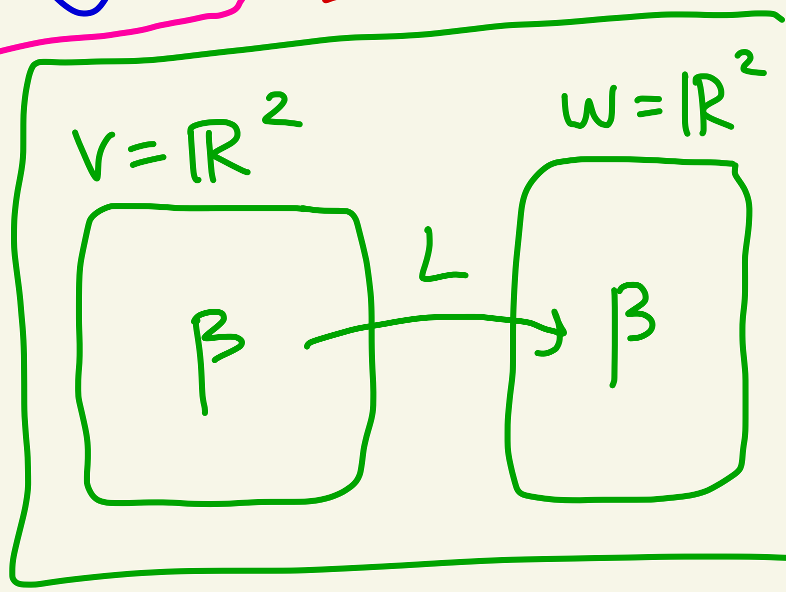
Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

You can check that L is a linear transformation.

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ ← standard basis for $\mathbb{R}^2$

Let's compute

$$[L]_\beta = [L]_\beta^\beta$$



$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

plug in basis for $V = \mathbb{R}^2$

Find the coordinates of output in terms of basis for $W = \mathbb{R}^2$

Thus,

$$[L]_\beta = \left(\left[L\begin{pmatrix}1\\0\end{pmatrix}\right]_\beta \;\middle|\; \left[L\begin{pmatrix}0\\1\end{pmatrix}\right]_\beta \right)$$

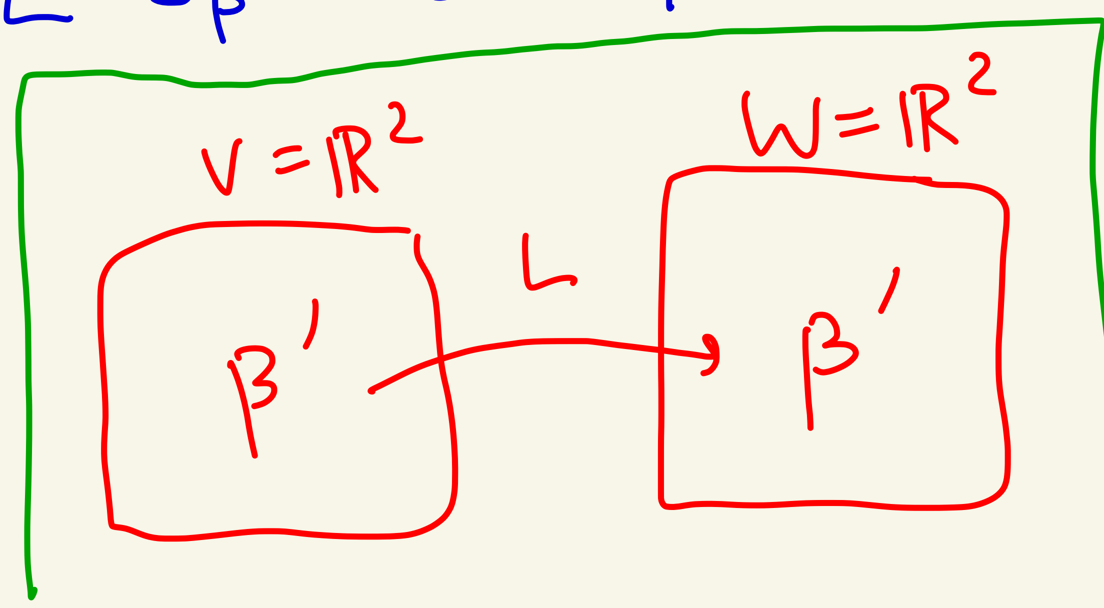
$$= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Let $\beta' = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right]$

$\boxed{\text{you can check that this is a basis for } \mathbb{R}^2}$

Let's find

$$[L]_{\beta'} = [L]_{\beta'}^{\beta'}$$



$$\boxed{\text{Recall } L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}}$$

$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = b \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

plug basis for $V = \mathbb{R}^2$ into L

write output in terms of basis for $W = \mathbb{R}^2$

This becomes

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a-c \\ a+c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} b-d \\ b+d \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} 2 &= a-c \\ 1 &= a+c \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} 0 &= b-d \\ -3 &= b+d \end{aligned}}$$

If you solve these you will get

$$a = \frac{3}{2}, \quad c = -\frac{1}{2}, \quad b = -\frac{3}{2}, \quad d = -\frac{3}{2}$$

So, $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$

$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underbrace{\frac{3}{2}}_{a} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \underbrace{\frac{1}{2}}_{c} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \underbrace{-\frac{3}{2}}_{b} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \underbrace{\frac{3}{2}}_{d} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus,

$$[L]_{\beta'} = [L]_{\beta'}^{\beta'}$$

$$= \left(\left[L\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{\beta'} \middle| \left[L\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]_{\beta'} \right)$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$$

Let's calculate $[L]_{\beta'}^{\beta}$

$$\underset{\beta'}{\mathbb{R}^2} \xrightarrow{L} \underset{\beta}{\mathbb{R}^2}$$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$$

$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

plug β' into L

write the answers in terms of β

$$[L]_{\beta'}^{\beta} = \left(\left[L\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\beta} \middle| \left[L\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]_{\beta} \right)$$

$$= \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$$

What do these matrices do?
Let's see with an example.

Pick $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ← these are the coordinates of v using the standard basis

$$\mathbb{R}^2 \ni \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{L} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \in \mathbb{R}^2$$

$$L\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2\cdot 1 - 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

The matrix that does the above is

$$[L]_\beta = [L]_\beta^\beta = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Let's see:

$$[L]_\beta \underbrace{[v]_\beta}_{v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 1+2 \\ 2-2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [L(v)]_\beta$$

$$L(v) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let's now see what $[L]_{\beta'} = [L]_{\beta'}^{\beta'}$ does to v.

We will show that

$$[L]_{\beta'} [v]_{\beta'} = [L(v)]_{\beta'}$$

So, $[L]_{\beta'} = [L]_{\beta'}^{\beta'}$ wants β' coordinates as its input and it computes L using the input and outputs the answer in β' coordinates.

What are v's β' coordinates?

Need to solve:

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underbrace{\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]}$$

This becomes

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_1 + \alpha_2 \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} 1 &= \alpha_1 - \alpha_2 \\ 2 &= \alpha_1 + \alpha_2 \end{aligned}}$$

adding gives
$3 = 2\alpha_1$
$\alpha_1 = 3/2$
So, $\alpha_2 = 2 - \alpha_1$
$= 2 - 3/2 = 1/2$

The solution is

$$\alpha_1 = 3/2$$
$$\alpha_2 = 1/2$$

So, $v = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Thus, $[v]_{\beta'} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

Then,

$$[L]_{\beta'}[v]_{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (3/2)(3/2) - (3/2)(1/2) \\ (-1/2)(3/2) - (3/2)(\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

This should be $[L(v)]_{\beta'}$.

Whose β' coordinates are these?

$$\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{2}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3/2 + 3/2\\3/2 - 3/2\end{pmatrix}$$

$$= \begin{pmatrix}6/4\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

$$= L(v)$$

So, $[L(v)]_{\beta'} = \begin{pmatrix}3/2\\-3/2\end{pmatrix}$.

Thus, $[L]_{\beta'}[v]_{\beta'} = [L(v)]_{\beta'}$

Now let's see what $[L]_{\beta'}^{\beta}$ does.

I claim that

$$[L]_{\beta'}^{\beta}[v]_{\beta'} = [L(v)]_{\beta}$$

So, $[L]_{\beta'}^{\beta}$ wants β' coordinates as input, and computes L, but gives the answer in β coordinates

We have that

$$[L]_{\beta}^{\beta'}[v]_{\beta'} = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (2)(3/2) + (0)(1/2) \\ (1)(3/2) + (-3)(1/2) \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [L(v)]_{\beta}$$

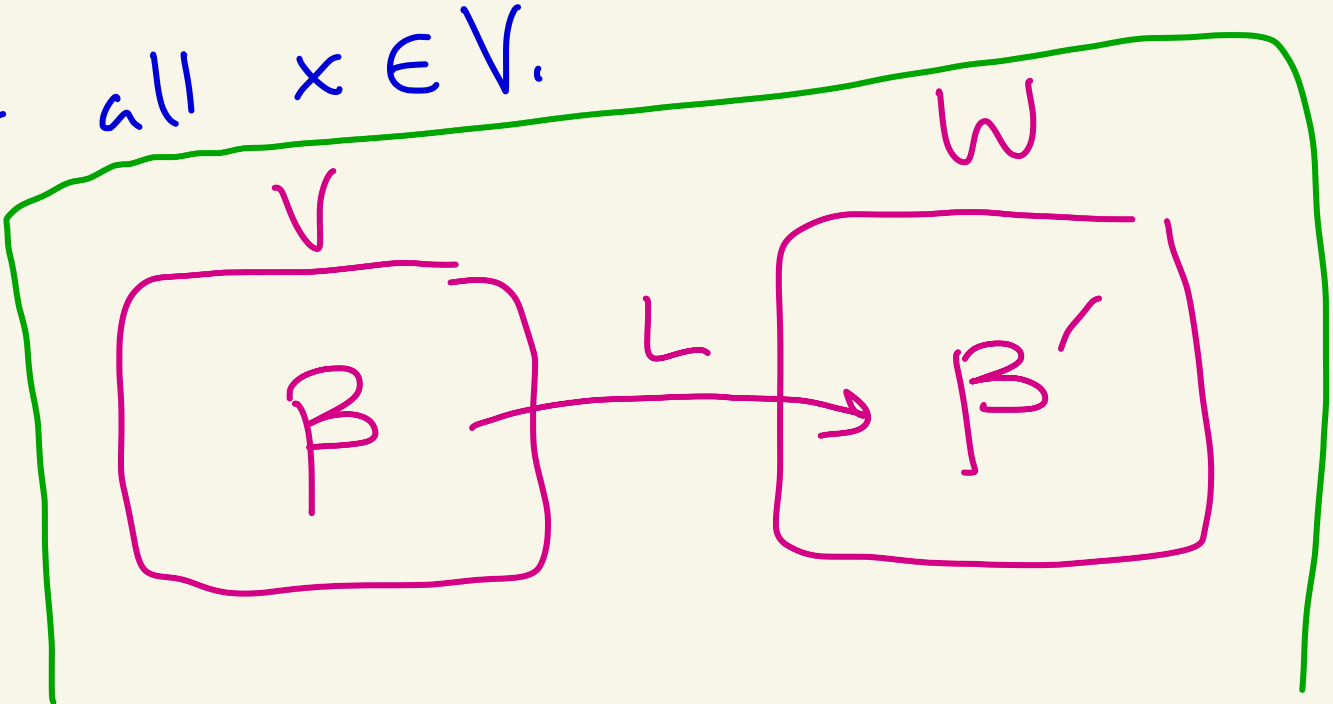
Theorem: Let V and W be finite-dimensional vector spaces over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V and $\beta' = [w_1, w_2, \ldots, w_m]$ be an ordered basis for W.

Let $L : V \to W$ be a linear transformation.

Then,

$$[L]_{\beta}^{\beta'} [x]_{\beta} = [L(x)]_{\beta'}$$

for all $x \in V$.



proof: Let $x \in V$.

Since $\beta = [v_1, v_2, \dots, v_n]$ is a basis for V, we may write

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

for some $\alpha_1, \alpha_2, \dots, \alpha_n \in F$.

Then, $[x]_\beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

Since $\beta' = [w_1, w_2, \dots, w_m]$ is a basis for W we may write

$$L(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$$

$$L(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$

$$\vdots \qquad\qquad \vdots$$

$$L(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$

where $a_{ij} \in F$.

Thus,

$$[L]_\beta^{\beta'} = \left([L(v_1)]_{\beta'} \middle| [L(v_2)]_{\beta'} \middle| \cdots \middle| [L(v_n)]_{\beta'} \right)$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Now let's get $[L(x)]_{\beta'}$ and show that

$$[L]_\beta^{\beta'}[x]_\beta = [L(x)]_{\beta'}$$

To get $[L(x)]_{\beta'}$ we need to express $L(x)$ in terms of β'!

We have that

$$
\begin{aligned}
L(x) &= L(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) \\
&\overset{\text{L is linear}}{=} \alpha_1 L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_n L(v_n) \\
&= \alpha_1 \Big(\overbrace{a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m}^{L(v_1)} \Big) \\
&\quad + \alpha_2 \Big(\underbrace{a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m}_{L(v_2)} \Big) \\
&\quad + \dots + \\
&\quad + \alpha_n \Big(\underbrace{a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m}_{L(v_n)} \Big) \\
&=
\end{aligned}
$$

$$= (\alpha_1 a_{11} + \alpha_2 a_{12} + \dots + \alpha_n a_{1n}) w_1$$
$$+ (\alpha_1 a_{21} + \alpha_2 a_{22} + \dots + \alpha_n a_{2n}) w_2$$
$$+ \dots +$$
$$+ (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \dots + \alpha_n a_{mn}) w_m$$

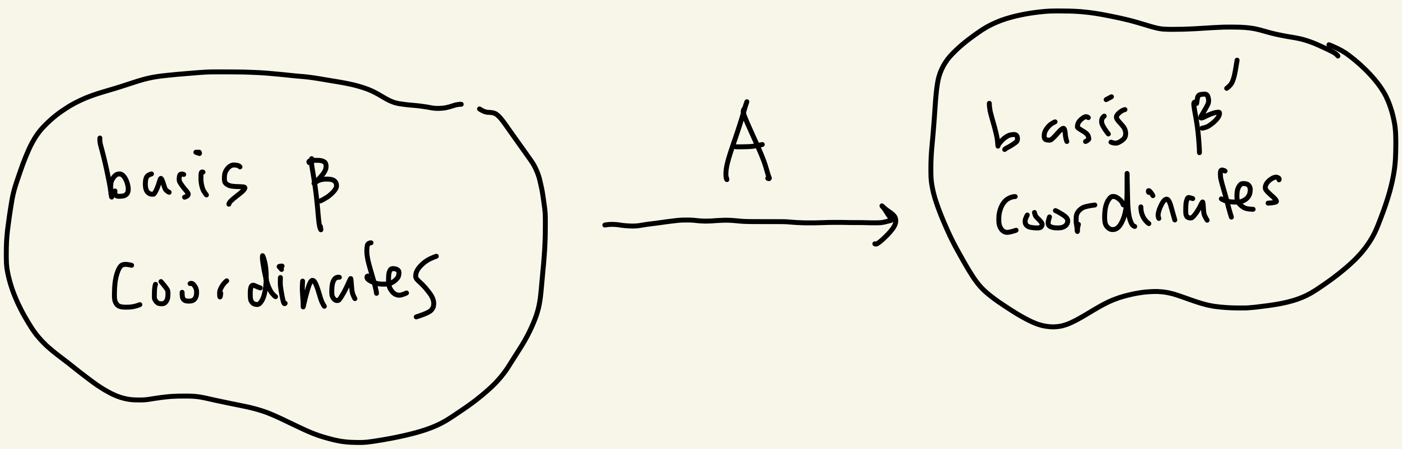
Thus,

$$[L(x)]_{\beta'} = \begin{pmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \dots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \dots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \dots \alpha_n a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

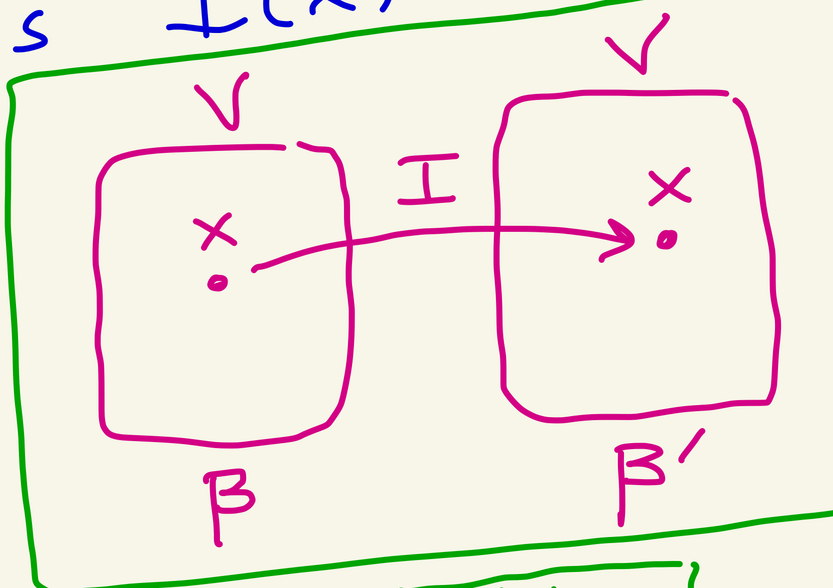
$$= [L]_{\beta}^{\beta'} [x]_{\beta} \qquad \blacksquare$$

We can use what we have developed to convert one coordinate system into another coordinate system.



We want a matrix that does this coordinate conversion.

Theorem: Let V be a finite-dimensional vector space over a field F. Let β and β' be two ordered bases for V. Let $I : V \to V$ be the identity function, that is $I(x) = x$ for all $x \in V$.



Then,

$$[I]_{\beta}^{\beta'}[x]_{\beta} = [x]_{\beta'}$$

Proof: We have that

$\boxed{I(x)=x}$

$$[I]_{\beta}^{\beta'}[x]_{\beta} = [I(x)]_{\beta'} = [x]_{\beta'}$$

$\boxed{\text{thm from Weds or pg 1 today}}$

$\blacksquare$

The matrix $[I]_{\beta}^{\beta'}$ is called the <u>change of basis matrix from β to β'.</u>

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ ← standard basis

and $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$ ← we used this basis last week

Lets calculate $[I]_{\beta}^{\beta'}$.

Recall
$I : \mathbb{R}^2 \to \mathbb{R}^2$
$I(v) = v$

We have that

$$I\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$I\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

plug β into I — express in terms of β'

This gives

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a-b \\ a+b \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c-d \\ c+d \end{pmatrix}$$

This becomes

$$\boxed{\begin{matrix} 1 = a - b \\ 0 = a + b \end{matrix}} \quad \text{and} \quad \boxed{\begin{matrix} 0 = c - d \\ 1 = c + d \end{matrix}}$$

If you solve these you will get

$a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = \frac{1}{2}$, $d = \frac{1}{2}$.

Thus,

$$[I]_{\beta}^{\beta'} = \left(\left[I \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_{\beta'} \middle| \left[I \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_{\beta'} \right)$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Let's test this matrix.

Pick $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ ← random vector we picked

$$[v]_\beta = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\boxed{v = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Then,

$$[v]_{\beta'} = \underbrace{[I]_\beta^{\beta'}}_{\text{this matrix turns } \beta\text{-coordinates into } \beta'\text{-coordinates}} [v]_\beta = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1}{2}\right)(2) + \left(\frac{1}{2}\right)(5) \\ \left(-\frac{1}{2}\right)(2) + \left(\frac{1}{2}\right)(5) \end{pmatrix} = \begin{pmatrix} 7/2 \\ 3/2 \end{pmatrix}$$

Checking: $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$

$$\frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/2 - 3/2 \\ 7/2 + 3/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = v$$

What about $w = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

Then, $w = -3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So, $[w]_\beta = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

And,

$$[w]_{\beta'} = [I]_\beta^{\beta'} [w]_\beta$$

$$= \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3/2 + 0 \\ 3/2 + 0 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} -3 \\ 0 \end{pmatrix} = w = -\frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

<u>Def:</u> Let V be a finite-dimensional vector space over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V.

So, $\dim(V) = n$. Define $\Phi : V \to F^n$ by $\Phi(x) = [x]_\beta$

[Note that Φ depends on the β that is chosen, so sometimes we will write Φ_β instead of just Φ]

We call Φ the <u>canonical isomorphism between V and F^n.</u>

Ex: $V = P_2(\mathbb{R})$, $F = \mathbb{R}$

Let $\beta = [1, x, x^2]$ ← standard basis

$\dim(P_2(\mathbb{R})) = 3$

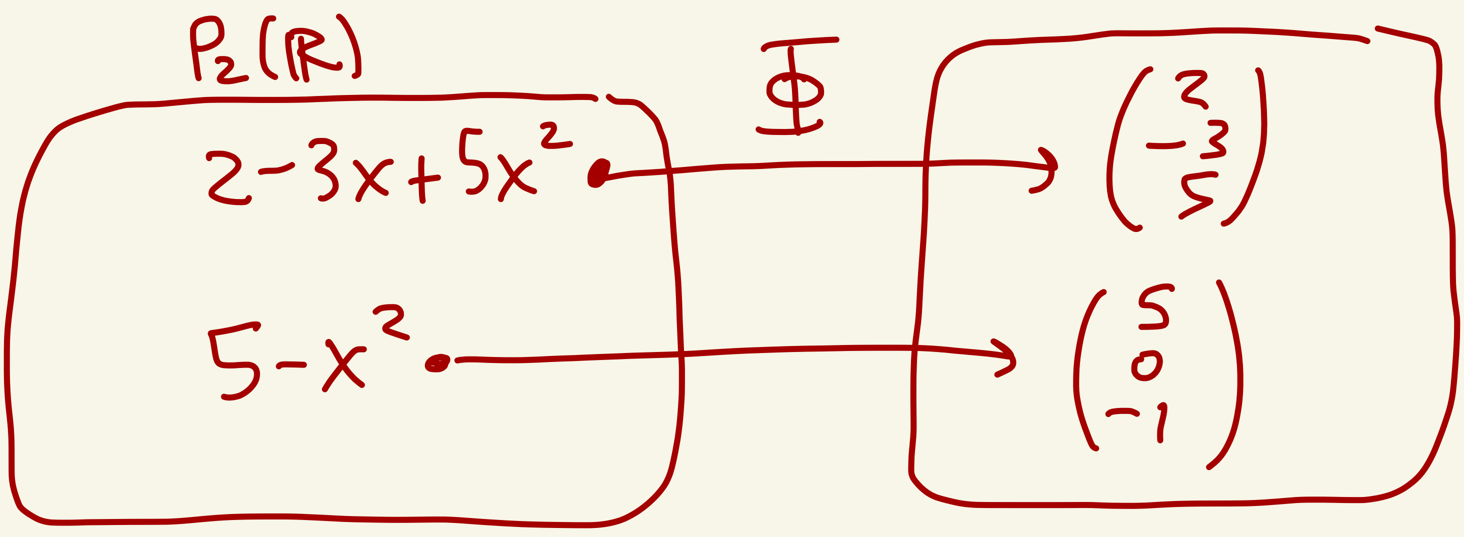
$\Phi : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$

Let $f_1 = 2 - 3x + 5x^2$

$$\Phi(f_1) = [f_1]_\beta = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

Let $f_2 = 5 - x^2$

$$\Phi(f_2) = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$



Note:

Do pages 33 - 36 if you have time. Otherwise, skip.

Let's show that Φ really is an isomorphism

Let V be a finite dimensional vector space over a field F.
Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V.
Pick the standard basis for F^n, ie

$$\beta' = \left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right]$$

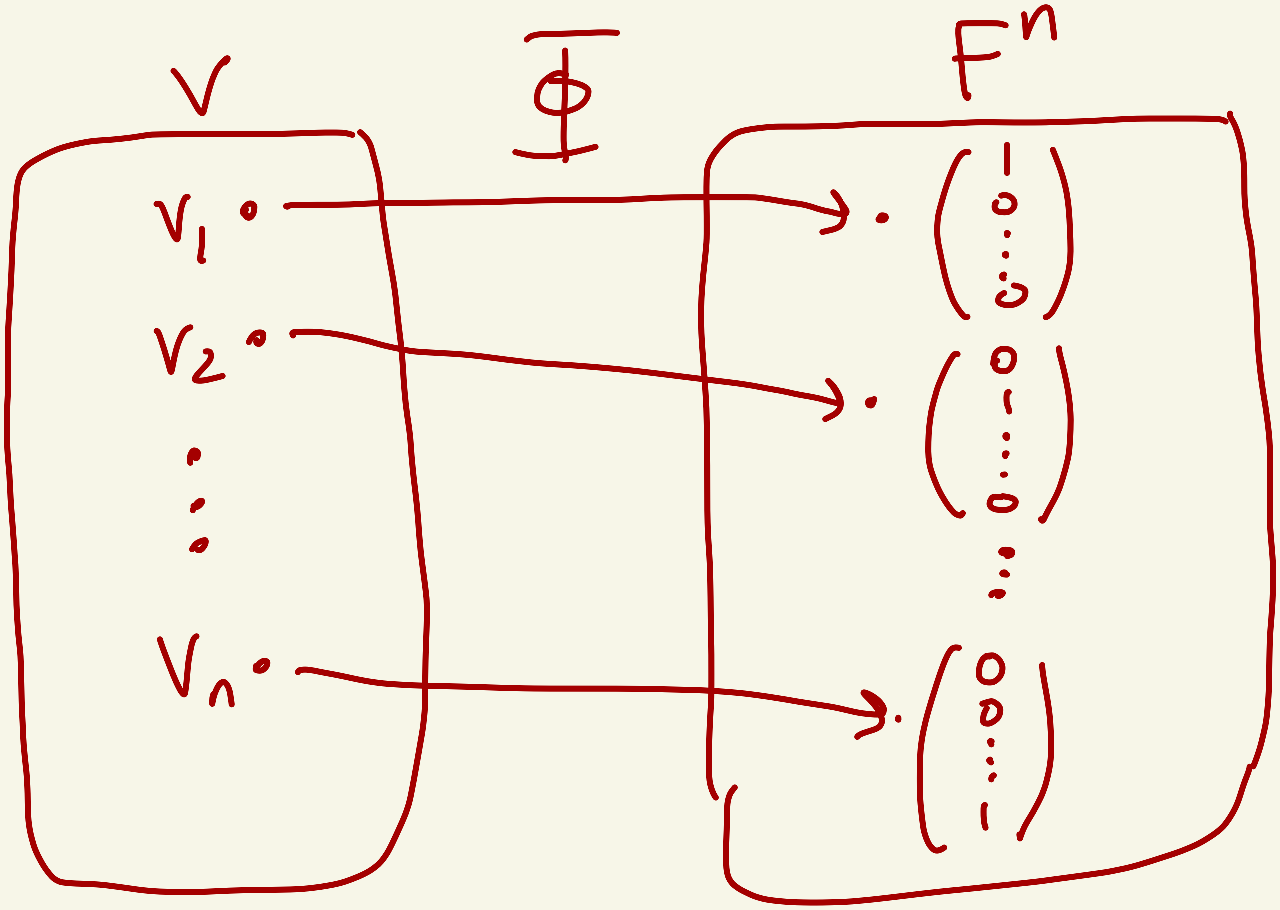
Then,

$$\Phi(v_1) = \Phi(1 \cdot v_1 + 0 v_2 + \cdots + 0 v_n) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Phi(v_2) = \Phi(0 \cdot v_1 + 1 \cdot v_2 + \cdots + 0 v_n) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi(v_n) = \Phi(0 v_1 + 0 v_2 + \cdots + 1 v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$



Also, if $x \in V$ and $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

then

$$\Phi(x) = \Phi(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

$$= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

This is the formula we had in a previous theorem about constructing linear transformations. It shows that Φ is a linear transformation.

The theorem also said that since

$$\{\Phi(v_1), \Phi(v_2), \ldots, \Phi(v_n)\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

is a basis for F^n

We have that Φ is an isomorphism between V and F^n.

Commutative diagram

Let V and W be finite-dimensional vector spaces over a field F.
Let $L : V \to W$ be a linear transformation.
Let β be an ordered basis for V and γ be an ordered basis for W.
Let $n = \dim(V)$ and $m = \dim(W)$.

$$
\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & L(x) \arrow[ddd, mapsto] \\
& V \arrow[r, "L"] \arrow[d, "\Phi_\beta"', "\cong"] & W \arrow[d, "\Phi_\gamma", "\cong"'] & \\
& F^n \arrow[r, "{[L]_\beta^\gamma}"] & F^m & \\
{[x]_\beta} \arrow[rrr, mapsto] & & & {[L]_\beta^\gamma [x]_\beta = [L(x)]_\gamma}
\end{tikzcd}
$$

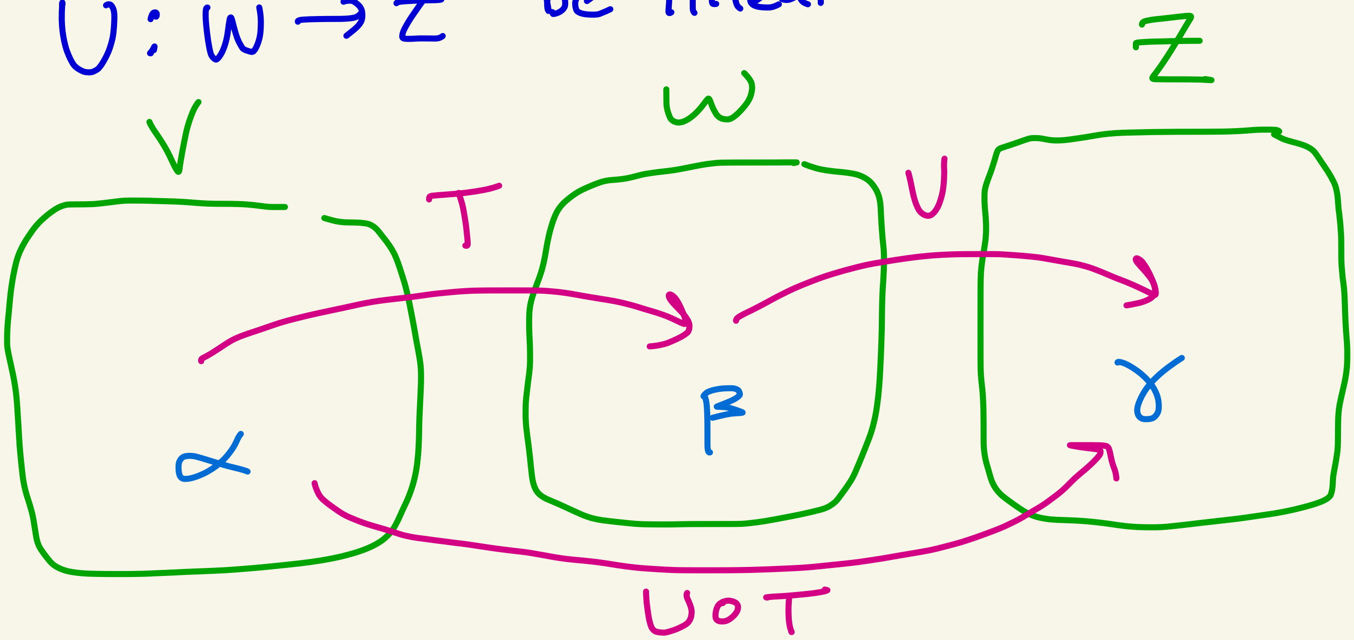
Note:

We are now back to what to cover after the pgs. 33-36 maybe.

On the next pages decide how much to cover by how much time is left. Definitely at least state the theorems on UoT and T^{-1}

HW 4

3. Let V, W, Z be finite-dimensional vector spaces over a field F with ordered bases α, β, γ respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations.



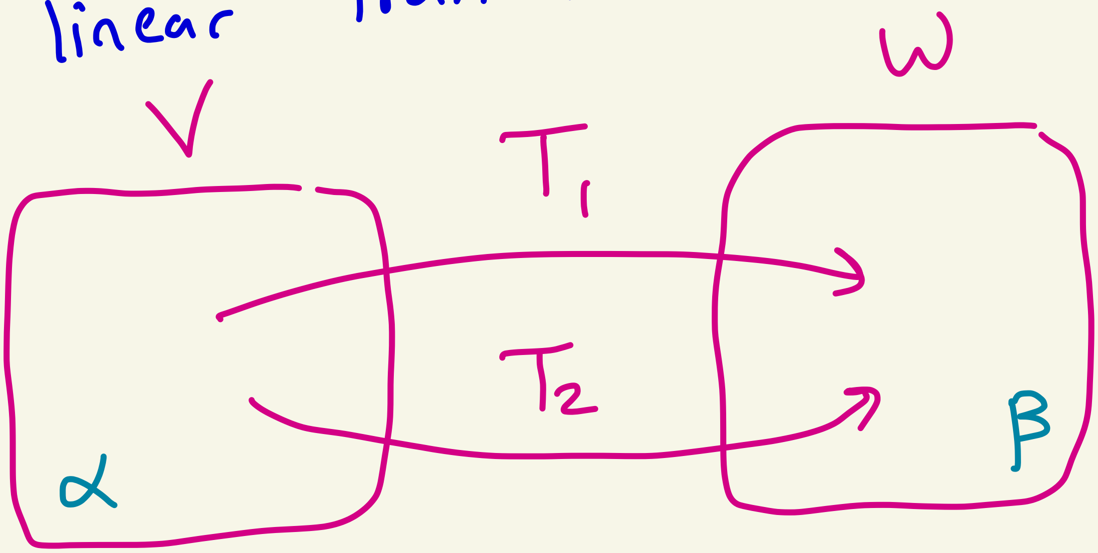
Then, $U \circ T : V \to Z$ is a linear transformation.

Also, $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

Proof: HW $\blacksquare$

HW 4

4 Let V and W be finite-dimensional vector spaces over a field F. Let α and β be ordered bases for V and W. Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear transformations.



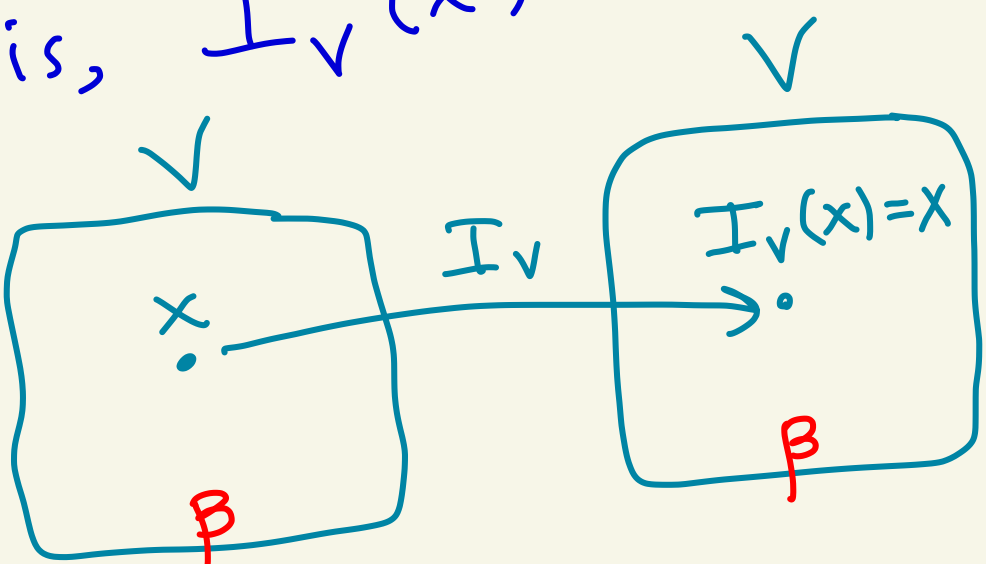
If $[T_1]_\alpha^\beta = [T_2]_\alpha^\beta$,

then $T_1 = T_2$.

HW 5

(2) Let V be a finite dimensional vector space over a field F. Let β be an ordered basis for V.

Let $I_V : V \to V$ be the identity linear transformation. That is, $I_V(x) = x$ for all $x \in V$.



Let $n = \dim(V)$.

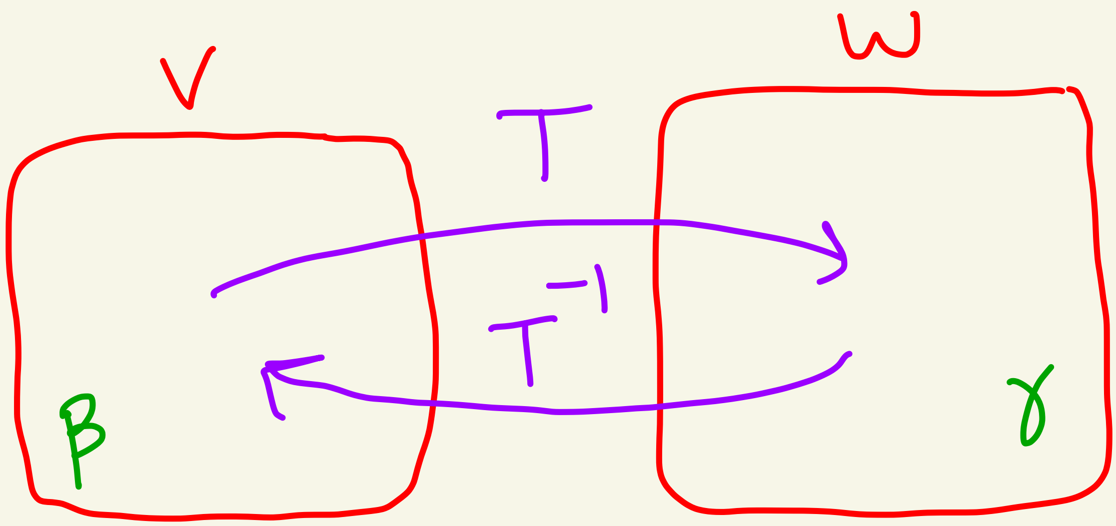
Then, $[I_V]_\beta = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

where I_n is the $n \times n$ identity matrix

Theorem: Let V and W be finite-dimensional vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Let β and γ be ordered bases for V and W, respectively.

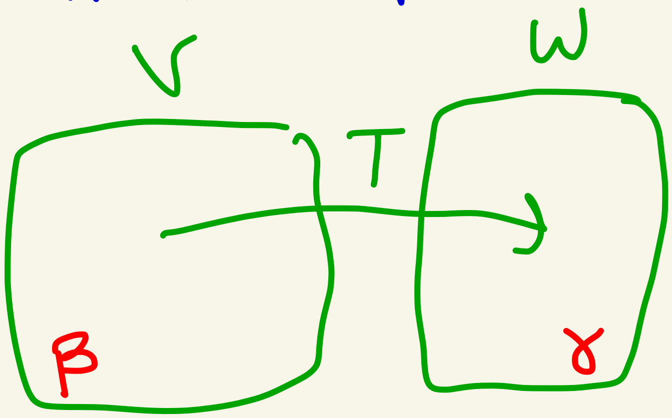
Then, T is an isomorphism (ie 1-1 and onto) iff $[T]_\beta^\gamma$ is invertible.

Furthermore, if this is the case then $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$



Proof:

($\Rightarrow$) Suppose T is an isomorphism.



Then T is one-to-one and onto, and from a theorem in class, $\dim(V) = \dim(W)$.

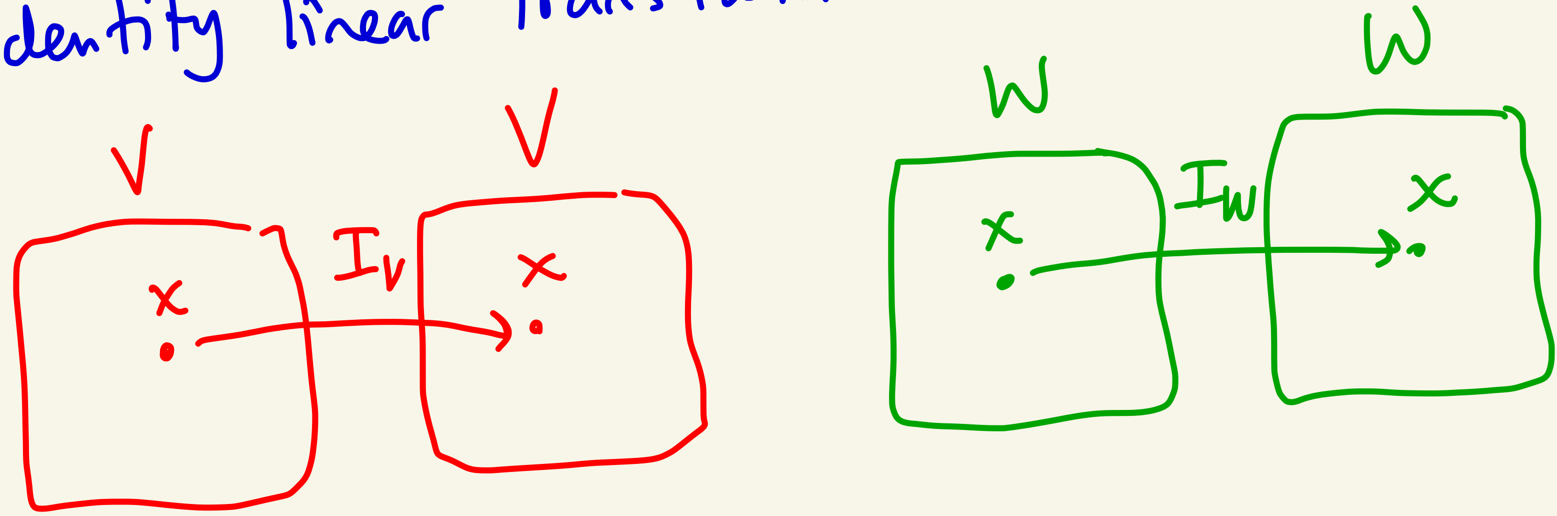
So, β and γ both have the same number of elements, lets say n elements each.

Then, $[T]_\beta^\gamma$ is $n \times n$.

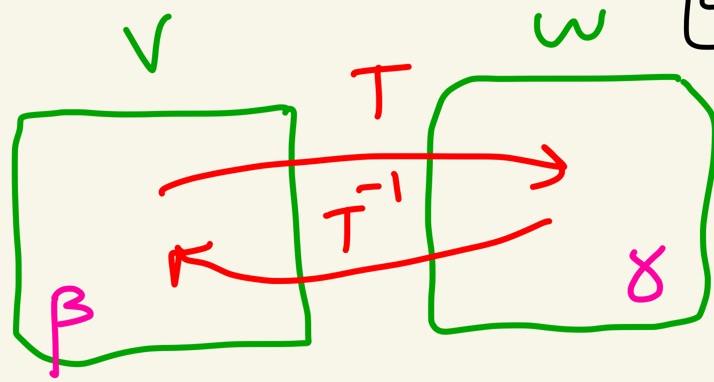
# of elements in β is the # of columns, # of elements in γ is the # of rows

Let $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ be the $n \times n$ identity matrix.

Let $I_V : V \to V$ be the identity linear transformation and $I_W : W \to W$ be the identity linear transformation



Because T is an isomorphism, $T^{-1} : W \to V$ exists and is a linear transformation (we did this in class).



Then,

$$[T^{-1}]_\gamma^\beta [T]_\beta^\gamma \overset{\text{HW 4}}{=} [T^{-1} \circ T]_\beta^\beta = [I_V]_\beta \overset{\text{HW 5}}{=} I_n$$

$$\boxed{T^{-1} \circ T = I_V}$$

and

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta \overset{\text{HW 4}}{=} [T \circ T^{-1}]_\gamma^\gamma = [I_W]_\gamma \overset{\text{HW 5}}{=} I_n$$

$$\boxed{T \circ T^{-1} = I_W}$$

Thus, $[T]_\beta^\gamma$ is invertible and

$$\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$$

($\Leftarrow$) Suppose that $[T]_\beta^\gamma$ is invertible.

We want to show that T is an isomorphism.

We will show that T^{-1} exists.

Since $[T]_\beta^\gamma$ is invertible it is a square matrix.

Let $A = [T]_\beta^\gamma$.

Suppose A is $n \times n$.

Then $\beta = [v_1, v_2, \ldots, v_n]$ and
$\gamma = [w_1, w_2, \ldots, w_n]$

where $v_1, \ldots, v_n \in V$
and $w_1, \ldots, w_n \in W$.

Let $B = A^{-1}$.

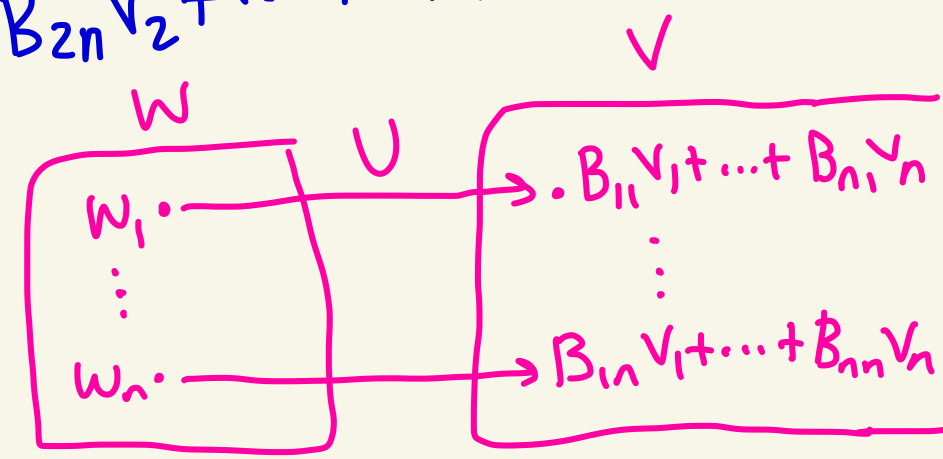
So, B is $n \times n$ also.

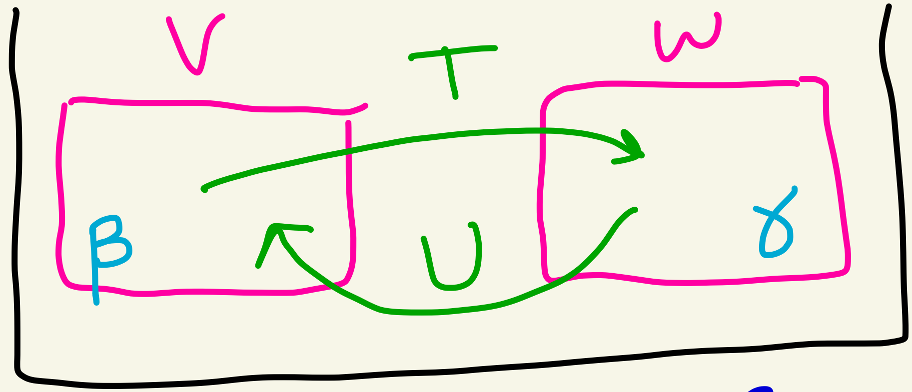
Let $B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}$

From a previous theorem in class we can construct a linear transformation $U : W \to V$ where

$$U(w_1) = B_{11} v_1 + B_{21} v_2 + \cdots + B_{n1} v_n$$
$$U(w_2) = B_{12} v_1 + B_{22} v_2 + \cdots + B_{n2} v_n$$
$$\vdots$$
$$U(w_n) = B_{1n} v_1 + B_{2n} v_2 + \cdots + B_{nn} v_n$$

So, $[U]_\gamma^\beta = B$





Then,

$$[U\circ T]_\beta = [U\circ T]_\beta^\beta = [U]_\gamma^\beta [T]_\beta^\gamma$$

HW 4

$$= BA = A^{-1}A = I_n = [I_V]_\beta$$

Since $[U\circ T]_\beta = [I_V]_\beta$, by HW

$$U\circ T = I_V.$$

Similarly,

$$[T\circ U]_\gamma = [T]_\beta^\gamma [U]_\gamma^\beta = AB = AA^{-1} = I_n = [I_W]_\gamma$$

So, by HW $T\circ U = I_W$.

Since $U \circ T = I_V$

and $T \circ U = I_W$

we know that $U = T^{-1}$.

So, $T^{-1} : W \to V$ exists

and T is 1-1 and onto.

$\blacksquare$

Corollary: Let V be a finite dimensional vector space over a field F. Let β and β' be ordered bases for V. Let $I: V \to V$ be the identity linear transformation [$I(x) = x$ for all $x \in V$]

Let $Q = [I]_{\beta}^{\beta'}$ be the change of basis matrix from β to β'.

Then:

1. Q is invertible and $Q^{-1} = [I]_{\beta'}^{\beta}$
2. If $T: V \to V$ is a linear transformation then

$$[T]_{\beta} = \underbrace{Q^{-1} [T]_{\beta'} Q}_{[I]_{\beta'}^{\beta} [T]_{\beta'} [I]_{\beta}^{\beta'}}$$

50

proof:

① I is invertible and $I^{-1} = I$.

[$I : V \to V \quad I(x) = x$ for all $x \in V$]

The theorem for Weds says that

$Q = [I]_{\beta}^{\beta'}$ is invertible and

$$Q^{-1} = \left([I]_{\beta}^{\beta'}\right)^{-1} = [I^{-1}]_{\beta'}^{\beta} = [I]_{\beta'}^{\beta}$$

② We have that

$$Q^{-1}[T]_{\beta'}Q = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}[I]_{\beta}^{\beta'}$$

$$= [I]_{\beta'}^{\beta}[T \circ I]_{\beta}^{\beta'}$$

$$= [I]_{\beta'}^{\beta}[T]_{\beta}^{\beta'} = [I \circ T]_{\beta}^{\beta}$$

$$= [T]_{\beta}^{\beta} = [T]_{\beta}$$ ▨

HW 4

$[U \circ T]_{\alpha}^{\gamma} = [U]_{\delta}^{\gamma}[T]_{\alpha}^{\delta}$

Def: Let A and B be $n \times n$ matrices with entries in a field F. We say that A and B are *similar* if there exists an $n \times n$ invertible matrix Q with entries from F where $B = Q^{-1} A Q$

In the previous theorem we saw that $[T]_\beta$ and $[T]_{\beta'}$ are similar matrices.

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let β be an ordered basis for V. Let $T: V \to V$ be a linear transformation.

Suppose $n = \dim(V)$.

If A is an $n \times n$ matrix with entries from F that is similar to $[T]_\beta$, then $A = [T]_\gamma$ where γ is some ordered basis for V.

<u>Proof:</u> We have $n = \dim(V)$.

Then $\beta = [v_1, v_2, \ldots, v_n]$ where $v_1, v_2, \ldots, v_n \in V$.

Also, $[T]_\beta$ is $n \times n$.

Since A is similar to $[T]_\beta$ we know that there exists an invertible matrix Q that is $n \times n$ and has entries in F and

$$A = Q^{-1} [T]_\beta Q$$

Let Q_{ij} denote the entry in Q in row i and column j.

That is,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & & & \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix}$$

Define the vectors $w_1, w_2, \ldots, w_n$ as follows:

$$\begin{aligned} w_1 &= Q_{11} v_1 + Q_{21} v_2 + \cdots + Q_{n1} v_n \\ w_2 &= Q_{12} v_1 + Q_{22} v_2 + \cdots + Q_{n2} v_n \\ &\vdots \\ w_n &= Q_{1n} v_1 + Q_{2n} v_2 + \cdots + Q_{nn} v_n \end{aligned}$$

So, $$w_j = \sum_{i=1}^{n} Q_{ij} v_i$$

this sum runs down the j-th column of Q

Let $\gamma = [w_1, w_2, \dots, w_n]$

We will now show that γ is a basis for V.

Let's show γ is a linearly <u>independent set</u>

Suppose

$$c_1 w_1 + c_2 w_2 + \dots + c_n w_n = \vec{0}$$

where $c_1, c_2, \dots, c_n \in F$.

Then,

$$c_1 \overbrace{(Q_{11}v_1 + Q_{21}v_2 + \dots + Q_{n1}v_n)}^{w_1}$$

$$+ \, c_2 \overbrace{(Q_{12}v_1 + Q_{22}v_2 + \dots + Q_{n2}v_n)}^{w_2}$$

$$+ \dots +$$

$$+ \, c_n \overbrace{(Q_{1n}v_1 + Q_{2n}v_2 + \dots + Q_{nn}v_n)}^{w_n}$$

$$= \vec{0}$$

Rearranging we get that

$$\left(c_1 Q_{11} + c_2 Q_{12} + \dots + c_n Q_{1n}\right) v_1$$
$$+ \left(c_1 Q_{21} + c_2 Q_{22} + \dots + c_n Q_{2n}\right) v_2$$
$$+ \dots +$$
$$+ \left(c_1 Q_{n1} + c_2 Q_{n2} + \dots + c_n Q_{nn}\right) v_n = \vec{0}$$

Since $\beta = [v_1, v_2, \dots, v_n]$ is a linearly independent set we have that

$$c_1 Q_{11} + c_2 Q_{12} + \dots + c_n Q_{1n} = 0$$
$$c_1 Q_{21} + c_2 Q_{22} + \dots + c_n Q_{2n} = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$c_1 Q_{n1} + c_2 Q_{n2} + \dots + c_n Q_{nn} = 0$$

Rewriting this as a matrix equation we get that

$$\begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus,

$$Q \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since Q is invertible, Q^{-1} exists and

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \underbrace{Q^{-1} Q}_{I_n} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus, $c_1 = 0$, $c_2 = 0$, ..., $c_n = 0$.

Thus $\gamma = [w_1, w_2, \ldots, w_n]$ is a linearly independent set.

Since γ contains n vectors and $\dim(V) = n$, we know γ is a basis for V.

By the definition of w_j, $Q = [I]_\gamma^\beta$

Why? $w_j = \sum_{i=1}^{n} Q_{ij} v_i$

$$I(w_j) = w_j = \sum_{i=1}^{n} Q_{ij} v_i$$

So the jth column of $[I]_\gamma^\beta$ is

$\begin{pmatrix} Q_{1j} \\ Q_{2j} \\ \vdots \\ Q_{nj} \end{pmatrix}$ which is the same as the jth column of Q.

Thus,

$$Q^{-1} = \left([I]_{\gamma}^{\beta}\right)^{-1} = [I^{-1}]_{\beta}^{\gamma} = [I]_{\beta}^{\gamma}$$

So,

$$A = Q^{-1}[T]_{\beta}Q$$
$$= [I]_{\beta}^{\gamma}[T]_{\beta}[I]_{\gamma}^{\beta}$$
$$= [T]_{\gamma}$$

$\blacksquare$