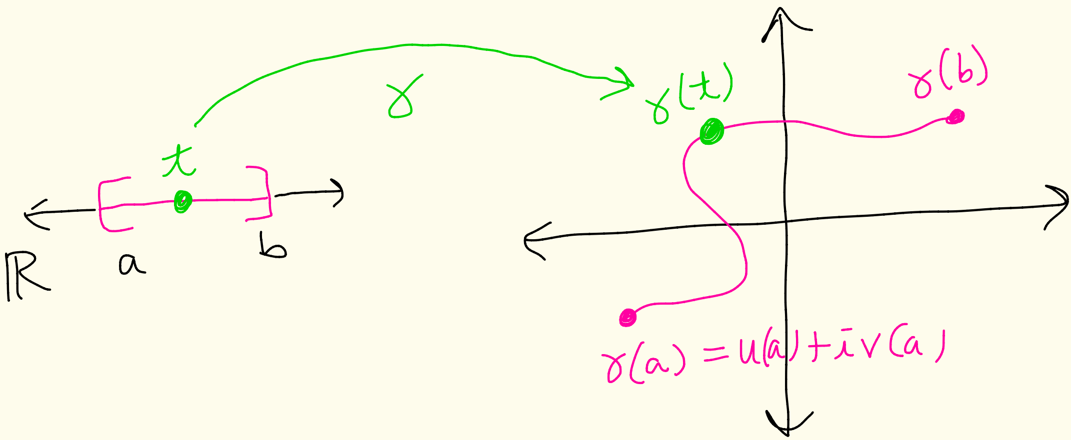
Topic 6/7 –

Contour integrals / Path-connected

$\boxed{\text{Curves}}$

<u>Def:</u> Let $a, b \in \mathbb{R}$ and $a < b$.
Let $\gamma : \underbrace{[a, b]}_{\text{in real \#s}} \longrightarrow \mathbb{C}$

So, $\gamma(t) = u(t) + i v(t)$
where u and v are real-valued functions defined on $[a, b]$.



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(def continued...)

- We say that γ is a <u>curve</u> (or <u>arc</u>) if u and v are continuous on $[a,b]$.

- If u' and v' exist on (a,b) then we define
$$\gamma'(t) = u'(t) + i v'(t)$$
and say that γ' exists and γ is <u>differentiable</u>.

- We say that γ is a <u>smooth curve</u> if γ is a curve, and γ is differentiable, and u' and v' are continuous on $[a,b]$.

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(def continued)

- γ is called piecewise-smooth if we can divide the interval $[a,b]$ into finitely many subintervals

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

such that γ is smooth on each $[a_i, a_{i+1}]$.

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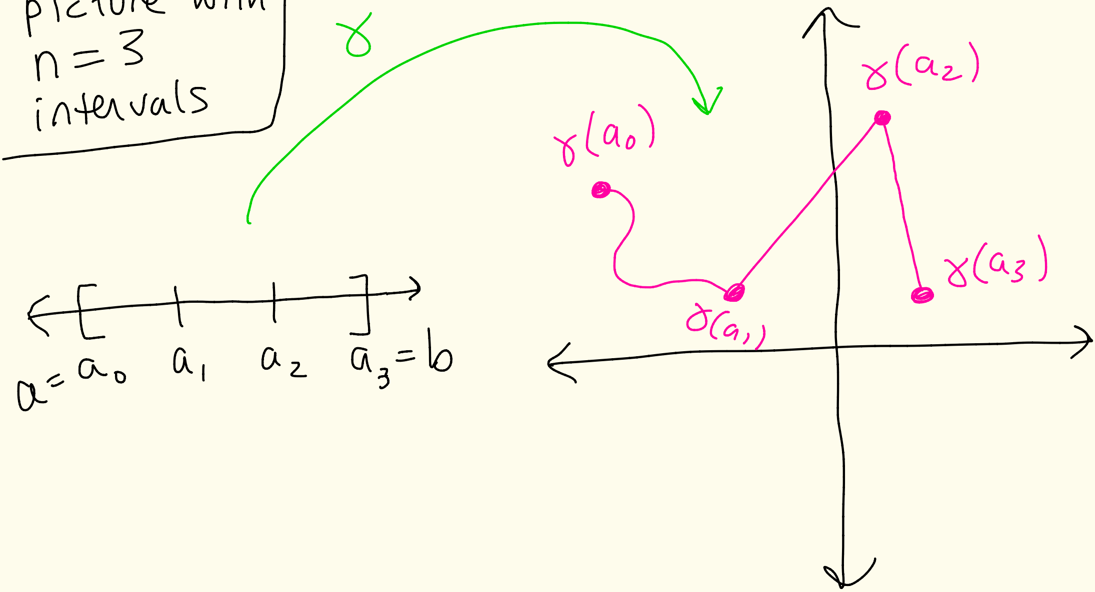
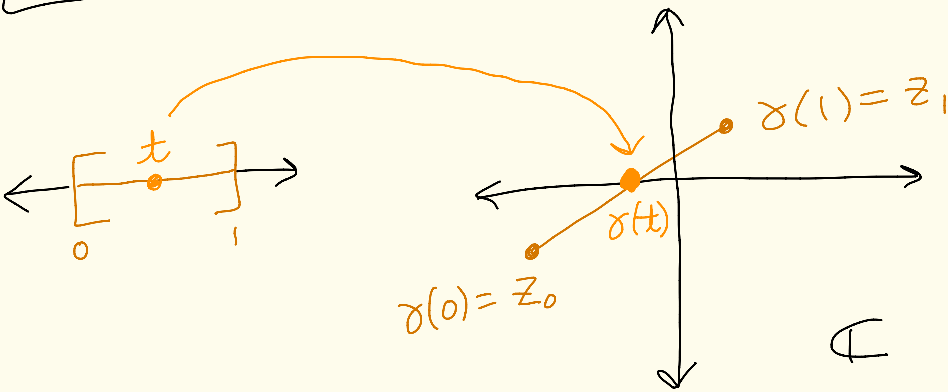
picture with $n=3$ intervals



Parameterizing straight lines

The line segment starting at z_0 and ending at z_1 can be parameterized as follows:

$$\boxed{\begin{aligned}&\gamma(t) = z_0 + t(z_1 - z_0)\\ &0 \le t \le 1\end{aligned}}$$



Ex:

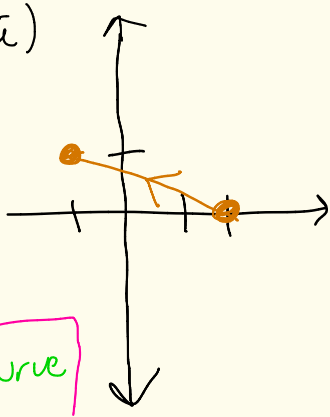
straight line between
$z_0 = 2$
$z_1 = -1 + i$

$$\gamma(t) = 2 + t(-3+i)$$
$$0 \le t \le 1$$

$$\gamma(t) = \underbrace{(2-3t)}_{u} + i\,\underbrace{t}_{v}$$

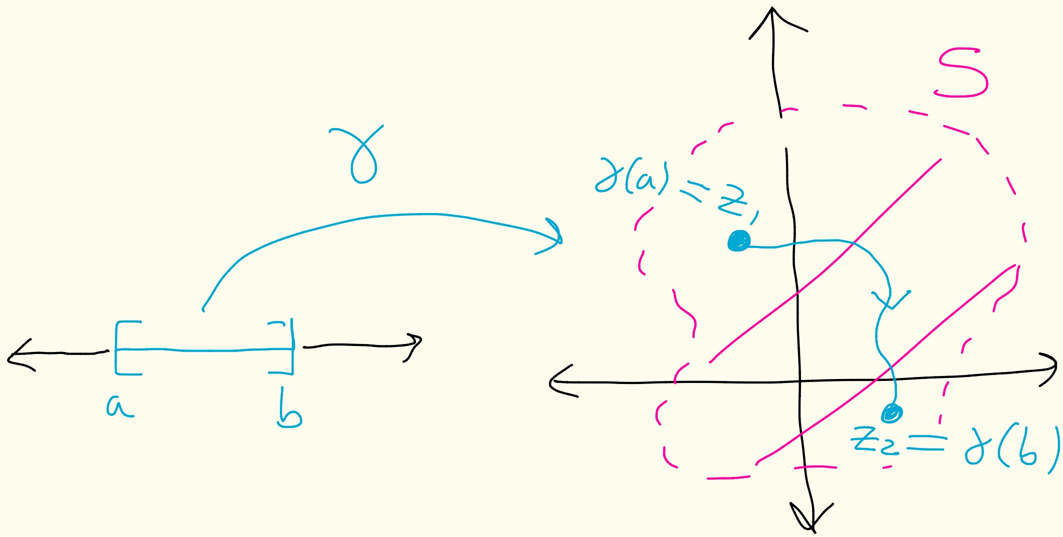
$u' = -3$
$v' = 1$

γ is a smooth curve



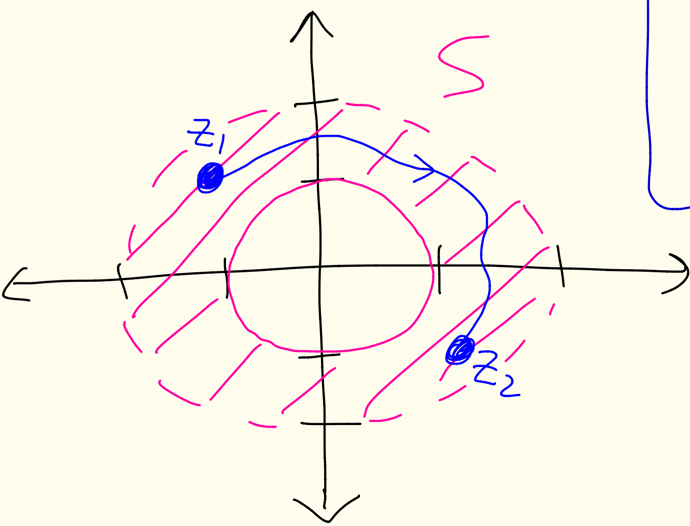
Some more topology

Def: A set $S \subseteq \mathbb{C}$ is called path-connected if for every pair of points $z_1, z_2 \in S$ there exists a piece-wise smooth curve $\gamma : [a,b] \longrightarrow S$ with $\gamma(a) = z_1$ and $\gamma(b) = z_2$.



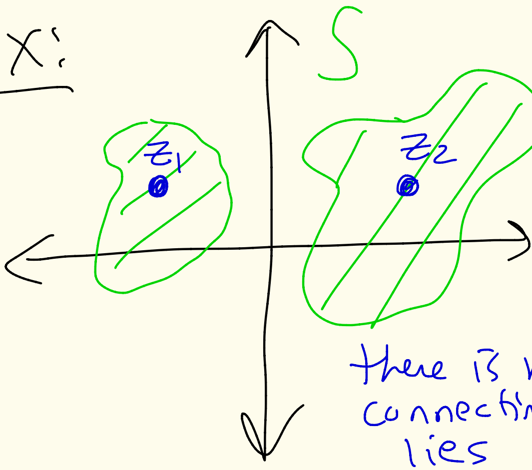
Ex:

$$S = \{ z \in \mathbb{C} \mid 1 \le |z| < 2 \}$$



S is path-connected

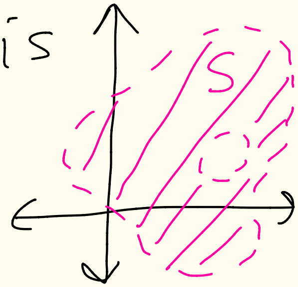
Ex:



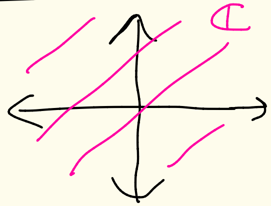
S is not path-connected

there is no smooth curve connecting z_1 to z_2 that lies in S.

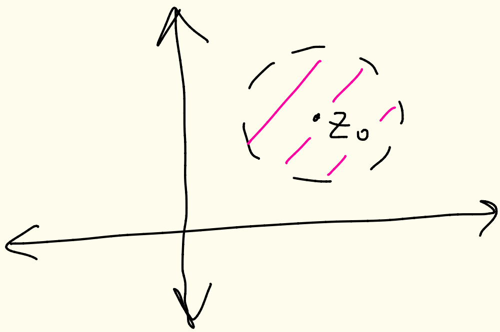
Def: Let $S \subseteq \mathbb{C}$.
If S is open and path-connected then we say that S is a **region** (or **domain**).



Ex: $\mathbb{C}$ is a region



Ex: $D(z_0; r)$ is a region.



Integrals

Def: Let $a, b \in \mathbb{R}$ and $a < b$. Let $h : [a,b] \to \mathbb{C}$ be a complex-valued function and let $h(t) = u(t) + i v(t)$. The *integral* of h on $[a,b]$ is defined to be

$$\int_a^b h(t)\,dt = \underbrace{\int_a^b u(t)\,dt}_{} + i \underbrace{\int_a^b v(t)\,dt}_{}$$

Calculus/real analysis integrals

Ex:

$$\int_0^2 \left[t^2 + i(t+1)\right] dt$$

$$\overset{\text{def}}{=} \left(\int_0^2 t^2\, dt\right) + i\left(\int_0^2 (t+1)\, dt\right)$$

$$= \frac{t^3}{3}\Big|_0^2 + i\left(\frac{t^2}{2} + t\right)\Big|_0^2$$

$$= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) + i\left(\left(\frac{2^2}{2} + 2\right) - \left(\frac{0^2}{2} + 0\right)\right)$$

$$= \boxed{\frac{8}{3} + 4i}$$

Integral of a function of a complex variable

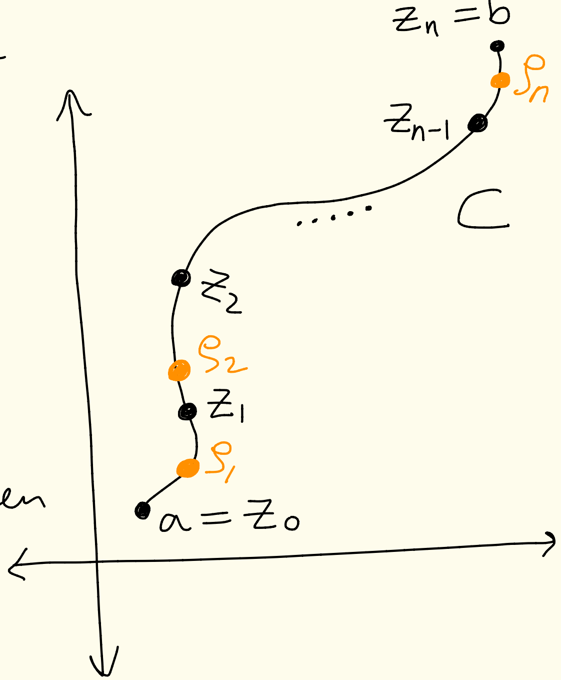
Let $f(z)$ be a function of a complex variable z.

Let C be piecewise smooth curve with endpoints a and b.

Suppose f is defined on all points on C.

Let $a = z_0$ and $b = z_n$.

Select $n-1$ distinct points on C between a and b, call them $z_1, z_2, \ldots, z_{n-1}$.



Let $\Delta z_k = z_k - z_{k-1}$.

Let ρ_k be any point on C between z_{k-1} and z_k.

We form the sum

$$\sum_{k=1}^{n} f(\rho_k)\,\Delta z_k$$

Now we further subdivide C letting n increase without bound and consider

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\varsigma_k)\,\Delta z_k$$

where n appoaches infinity and $|\Delta z_k|$ approaches zero for all k.

If this limit exists and is independent of the particular set of subdivisions used, we define its value to be the <u>definite integral of f along C</u>, that is

$$\int_C f(z)\,dz = \lim_{\substack{n\to\infty \\ \max|\Delta z_k|\to 0}} \sum_{k=1}^{n} f(\varsigma_k)\,\Delta z_k$$

Theorem: Suppose that $f: A \to \mathbb{C}$ is continuous on an open set $A \subseteq \mathbb{C}$ and let $\gamma:[a,b] \to \mathbb{C}$ be a piecewise smooth curve that lies in A. Then $\int_\gamma f$ exists. Furthermore, if the partition of $[a,b]$ that makes γ piecewise is $a = a_0 < a_1 < \cdots < a_n = b$ then

$$\int_\gamma f = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)\,dt$$

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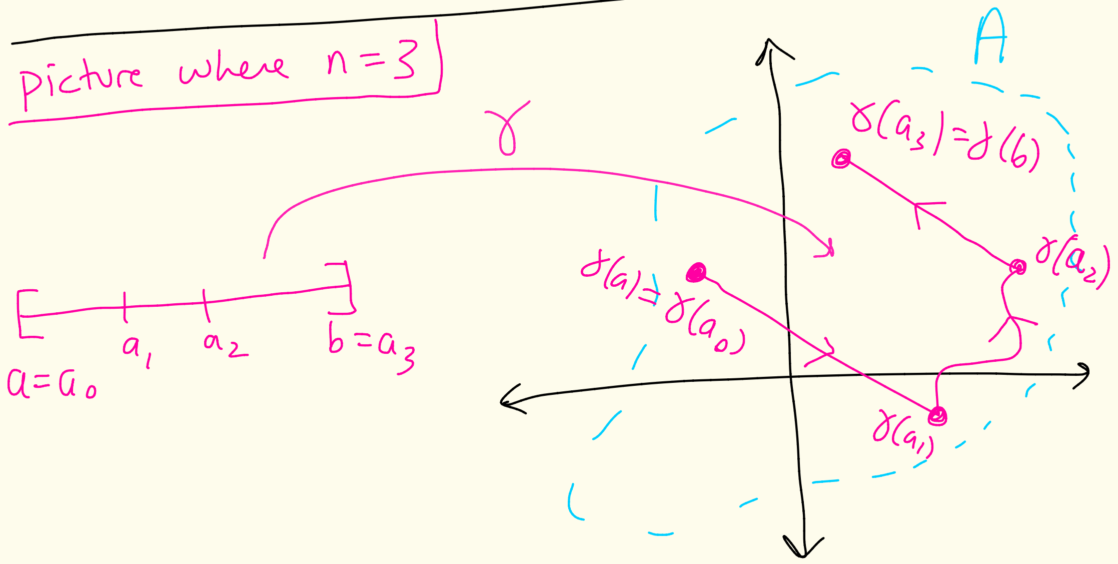
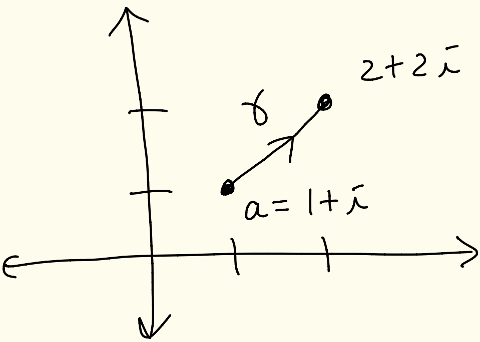
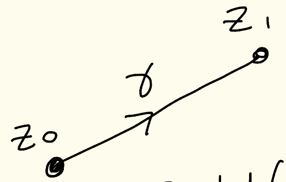
picture where $n=3$



Ex: Integrate $f(z) = 2z+1$ on the line segment starting at $1+i$ and ending at $2+2i$.



Formula for line



$$\gamma(t) = z_0 + t(z_1 - z_0)$$
$$0 \le t \le 1$$

$$\gamma(t) = (1+i) + t\left[(2+2i) - (1+i)\right], \quad 0 \le t \le 1$$

$$\gamma(t) = (1+i) + t\left[1+i\right], \quad 0 \le t \le 1$$

$$\gamma(t) = (1+t) + i(1+t), \quad 0 \le t \le 1$$

$$\gamma'(t) = 1 + i(1) = 1+i, \quad 0 \le t \le 1$$

$$\boxed{\begin{aligned} \gamma &= u + iv \\ \gamma' &= u' + iv' \end{aligned}}$$

$$\int_\gamma f = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_0^1 \underbrace{\left\{2\left[(1+t) + i(1+t)\right] + 1\right\}}_{f(\gamma(t))} \underbrace{\left[(1+i)\right]}_{\gamma'(t)} \underbrace{dt}_{dt}$$

$$\int_0^1 \left\{ 2\left[(1+t)+i(1+t)\right] + 1 \right\} \left[(1+i)\right] dt$$

$$= \int_0^1 \left(\underbrace{2+2t+2i+2it+1}_{\boxed{3+2t+2i+2it}}\right)(1+i)\,dt$$

$$= \int_0^1 \left[(3+2t+2i+2it) + (3i+2ti-2-2t)\right] dt$$

$$= \int_0^1 \left[1+i(5+4t)\right] dt$$

$$= \left(\int_0^1 1\,dt\right) + i\left(\int_0^1 (5+4t)\,dt\right)$$

$$= \left(t\Big|_0^1\right) + i\left(5t+4\frac{t^2}{2}\right)\Bigg|_0^1$$

$$= (1-0) + i\left((5+2)-0\right)$$

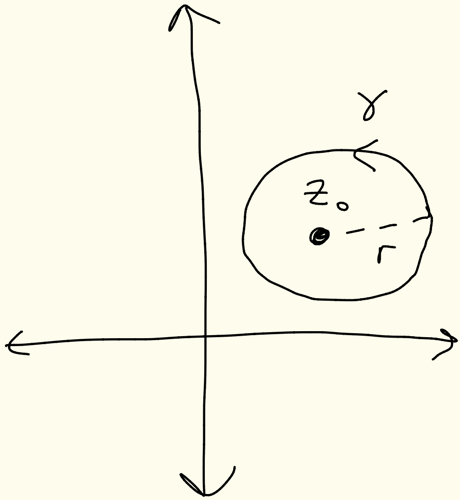
$$= 1+7i$$

<u>Def:</u>

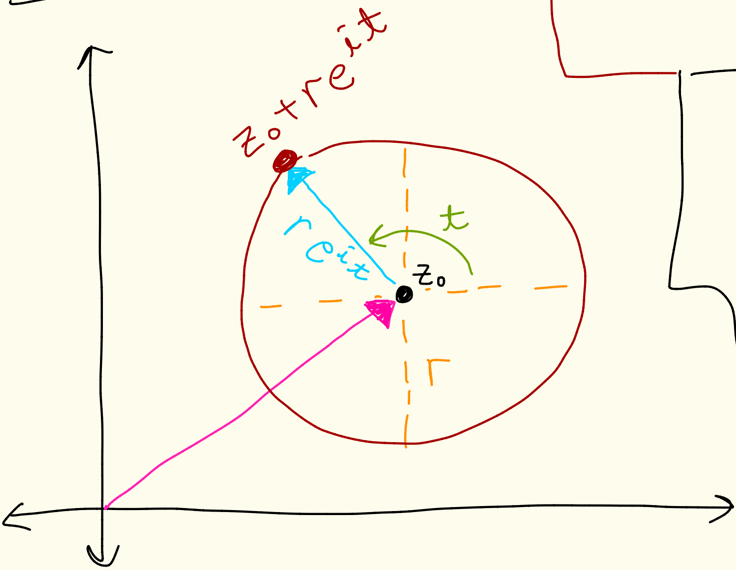
$$\int_a^b \left[u(t)+iv(t)\right] dt = \left(\int_a^b u(t)\,dt\right) + i\left(\int_a^b v(t)\,dt\right)$$

Formula for a circle centered at z_0 with radius r, going around it once counter-clockwise:

$$\boxed{\begin{aligned}&\gamma(t) = z_0 + re^{it}\\ &0 \le t \le 2\pi\end{aligned}}$$



Idea:

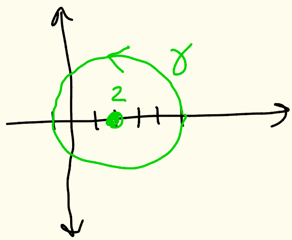


Ex:

$z_0 = 2$

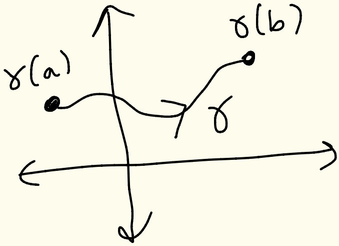
$r = 3$

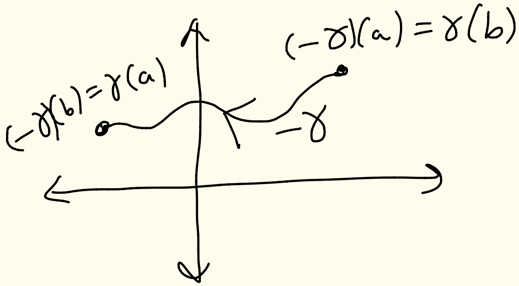
$$\gamma(t) = 2 + 3e^{it}$$
$$0 \le t \le 2\pi$$



Def: For a curve $\gamma : [a,b] \to \mathbb{C}$ we define the opposite curve, $-\gamma : [a,b] \to \mathbb{C}$, by setting

$$(-\gamma)(t) = \gamma(a+b-t)$$

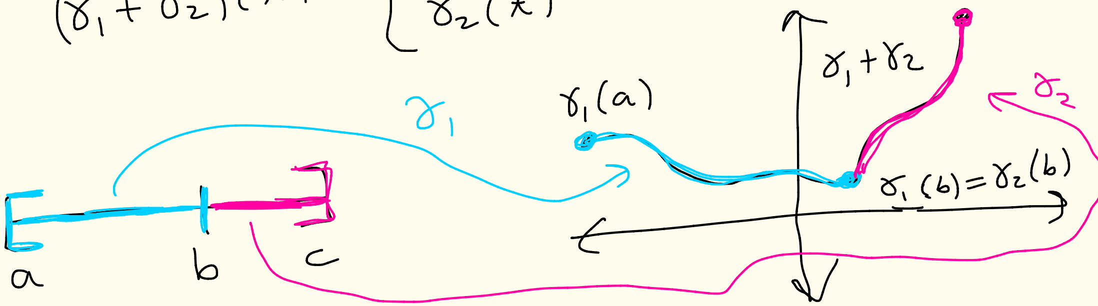




$-\gamma$ reverses the direction of γ

Def: Suppose $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ are two curves with $\gamma_1(b) = \gamma_2(b)$. The sum, or join, or union, $\gamma_1 + \gamma_2$, is defined to be $(\gamma_1+\gamma_2) : [a,c] \to \mathbb{C}$ where

$$(\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(t) & a \le t \le b \\ \gamma_2(t) & b \le t \le c \end{cases}$$



<u>Theorem</u>: Let $c_1, c_2 \in \mathbb{C}$.
Let f and g be continuous functions on an open set containing the piecewise smooth curves $\gamma_1, \gamma_2, \gamma$.

Then:

① $$\int_{\gamma} (c_1 f + c_2 g) = c_1 \int_{\gamma} f + c_2 \int_{\gamma} g$$

② $$\int_{-\gamma} f = -\int_{\gamma} f$$

reversing directions introduces a minus sign

③ $$\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

Theorem: If γ and $\tilde{\gamma}$ are "parameterizations" of the same curve, then $\int_\gamma f = \int_{\tilde{\gamma}} f$.

Roughly, same parameterization of a curve means they trace out the same curve but just not at the same speed.

Ex:

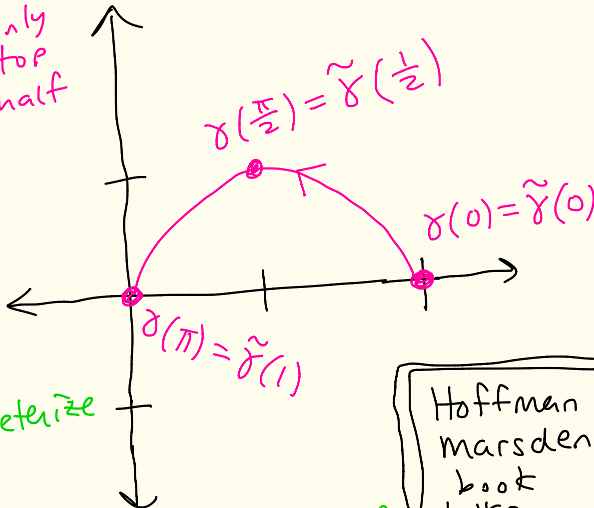
center is 1 radius is 1

$$\gamma(t) = 1 + e^{it}$$
$$0 \le t \le \pi$$

only top half

$$\tilde{\gamma}(t) = 1 + e^{i\pi t}$$
$$0 \le t \le 1$$

γ and $\tilde{\gamma}$ both trace out



Both γ and $\tilde{\gamma}$ parameterize the same curve so

$\int_\gamma f = \int_{\tilde{\gamma}} f$ for any continuous f.

Hoffman marsden book talks more about this

Def: (Arc length of a curve)

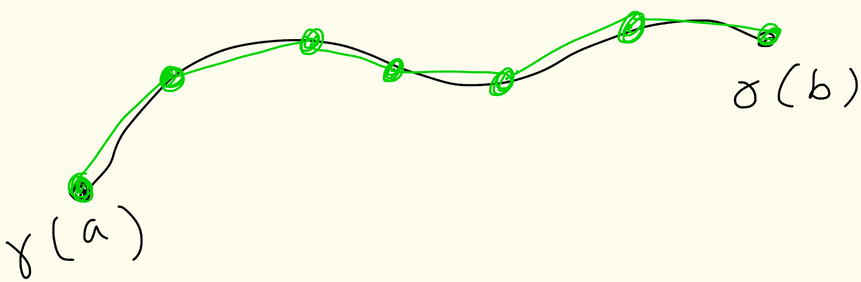
Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve, where $\gamma(t) = u(t) + iv(t)$.

The arclength of γ is defined to be

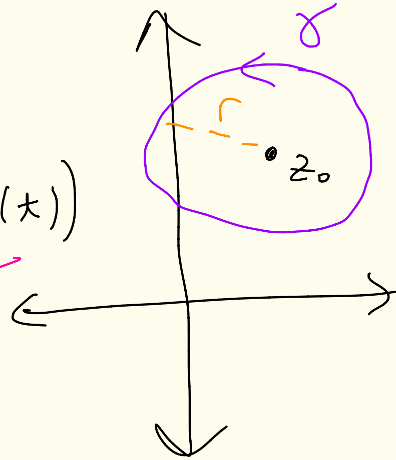
$$\text{arclength}(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

$$= \int_a^b \sqrt{(u'(t))^2 + (v'(t))^2}\, dt$$

If γ is piecewise smooth, its arclength is the sum of the arclengths of its smooth components.



Ex: Consider $\gamma(t) = z_0 + r e^{it}$ where $0 \le t \le 2\pi$ be the circle of radius r centered at $z_0 = x_0 + i y_0$



$$\gamma(t) = z_0 + r e^{it}$$

$$= \underbrace{\left(x_0 + r\cos(t)\right)}_{u(t)} + i \underbrace{\left(y_0 + r\sin(t)\right)}_{v(t)}$$

$$u(t) + i\, v(t)$$

$$u'(t) = -r\sin(t)$$

$$v'(t) = r\cos(t)$$

$$\text{arclength}(\gamma) = \int_0^{2\pi} \sqrt{\left(-r\sin(t)\right)^2 + \left(r\cos(t)\right)^2}\, dt$$

$$= \int_0^{2\pi} r\sqrt{\cos^2(t) + \sin^2(t)}\, dt$$

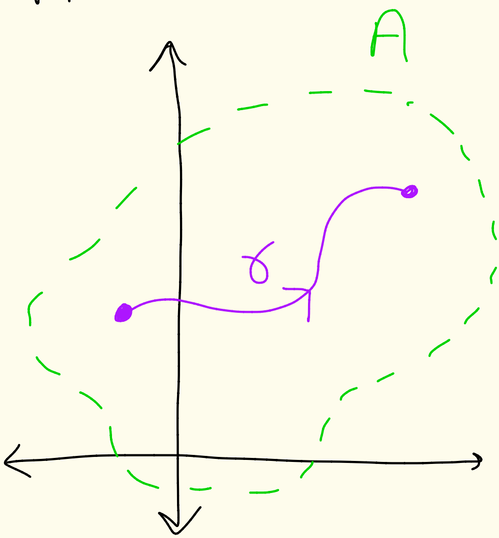
$$= \int_0^{2\pi} r\, dt = rt \Big|_0^{2\pi} = 2\pi r$$

circumference of circle

<u>Theorem:</u> Let $f: A \to \mathbb{C}$ where A is an open set. Suppose f is continuous on A. Let γ be a piecewise-smooth curve in A. Suppose that

$$|f(z)| \leq M$$

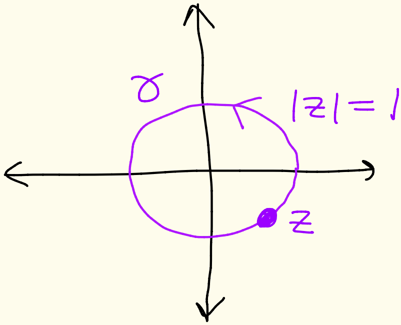
for all z on γ where $M \geq 0$ is a real number.



Then, $\displaystyle \left| \int_\gamma f \right| \leq M \left[\text{arclength}(\gamma) \right]$

Ex: Let γ be the unit circle oriented counter-clockwise.

Let $f(z) = z^2 + 2z + 5$.



Suppose z is on γ.

Then $|z| = 1$ and

$$|f(z)| = |z^2 + 2z + 5|$$

$$\leq |z^2| + |2z| + |5|$$

Δ-inequality

$$= |z|^2 + 2|z| + 5$$

$$= 1^2 + 2(1) + 5 = 8$$

z on γ
|z| = 1

Let $M = 8$.

Then, when z is on γ, $|f(z)| \leq M = 8$.

f is continuous everywhere in $\mathbb{C}$, so we can use the theorem. So,

$$\left| \int_\gamma f \right| \leq 8 \cdot \text{arclength}(\gamma) = 8 \cdot (2\pi) = 16\pi$$

$\boxed{\text{You could calculate that } \int_\gamma f = 0 \text{ so this bound isn't very good}}$

proof of theorem:

We wish to bound $\left| \int_\gamma f(z)\,dz \right|$.

We are assuming $|f(z)| \le M$ for all z on γ

For simplicity, we begin by assuming that γ is a smooth curve.

Then $\int_\gamma f(z)\,dz = \boxed{\int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$

for some $a<b$.

equal to

Let $g(t) = f(\gamma(t))\,\gamma'(t)$.

Then, $\boxed{\int_a^b g(t)\,dt} = re^{i\theta}$ for some r, θ, $r \ge 0$.

Then,

$$r = e^{-i\theta}\int_a^b g(t)\,dt = \int_a^b e^{-i\theta} g(t)\,dt$$

So,

$$r = \mathrm{Re}(r) = \mathrm{Re}\left(\int_a^b e^{-i\theta} g(t)\,dt\right)$$

$$\overset{(*)}{=} \int_a^b \mathrm{Re}\left(e^{-i\theta} g(t)\right) dt$$

$\boxed{\begin{array}{l} (*) \text{ is because} \\ \mathrm{Re}\left(\int_a^b u(t) + i v(t)\,dt\right) = \mathrm{Re}\left(\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\right) \\ = \int_a^b u(t)\,dt = \int_a^b \mathrm{Re}(u(t) + i v(t))\,dt \end{array}}$

From class, $\mathrm{Re}(z) \le |\mathrm{Re}(z)| \le |z|$
for all $z \in \mathbb{C}$.

Thus, $\mathrm{Re}(e^{-i\theta} g(t)) \le |e^{-i\theta} g(t)|$

$$= \underbrace{|e^{-i\theta}|}_{1}\,|g(t)| = |g(t)|$$

So, $\underbrace{\int_a^b \mathrm{Re}(e^{-i\theta} g(t))\,dt \le \int_a^b |g(t)|\,dt}_{\text{normal calculus integrals here}}$

Therefore,

$$\left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| = \left|\int_a^b g(t)\,dt\right|$$

$$= |re^{i\theta}| = |r|\underbrace{|e^{-i\theta}|}_{1} = |r|$$

$$= r = \int_a^b \mathrm{Re}\left(e^{-i\theta} g(t)\right) dt$$

$$\le \int_a^b |g(t)|\,dt = \int_a^b |f(\gamma(t))\cdot\gamma'(t)|\,dt$$

$$= \int_a^b \underbrace{|f(\gamma(t))|}_{\le M}\,|\gamma'(t)|\,dt$$

calculus integrals so can do bound

$$\le \int_a^b M\,|\gamma'(t)|\,dt = M\int_a^b |\gamma'(t)|\,dt$$

$$= M \text{ arclength}(\gamma)$$

$\gamma(t) = u(t) + iv(t)$

$|\gamma'(t)| = \sqrt{(u'(t))^2 + (v'(t))^2}$

γ smooth curve

Suppose now that γ is piece-wise smooth.

Then $\gamma = \sum_{i=1}^{n} \gamma_i$ where each γ_i is smooth and γ_i ends where γ_{i+1} begins.

Then,

$$\left|\int_\gamma f\right| = \left|\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f\right|$$

$$= \left|\int_{\gamma_1} f + \int_{\gamma_2} f + \cdots + \int_{\gamma_n} f\right|$$

$$\overset{\Delta}{\leq} \left|\int_{\gamma_1} f\right| + \left|\int_{\gamma_2} f\right| + \cdots + \left|\int_{\gamma_n} f\right|$$

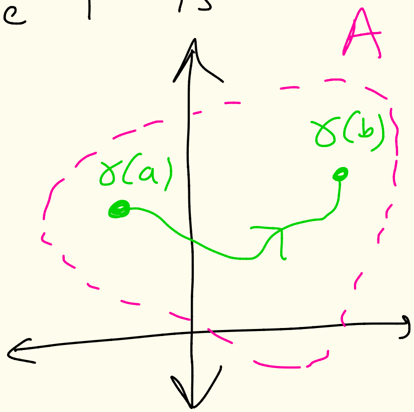
$$\leq M \cdot \text{arclength}(\gamma_1) + M\,\text{arclength}(\gamma_2) + \cdots + M\,\text{arclength}(\gamma_n)$$

$$= M \sum_{i=1}^{n} \text{arclength}(\gamma_i) = M\,\text{arclength}(\gamma).$$

So, $\left|\int_\gamma f\right| \leq M\,\text{arclength}(\gamma)$. $\blacksquare$

Fundametal Theorem of Calculus

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a piecewise smooth curve and that F is a function defined and analytic on an open set A containing γ. Assume F' is continuous in A.



Then,

$$\int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

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proof: Break $[a,b]$ into subintervals $[a_i, a_{i+1}]$ where γ' exists on (a_i, a_{i+1}) and is continuous on $[a_i, a_{i+1}]$ where $a_0 = a$ and $a_n = b$.

Suppose $F(\gamma(t)) = u(t) + i v(t)$.

Then, differentiating both sides gives

$$F'(\gamma(t)) \cdot \gamma'(t) = u'(t) + i v'(t).$$

So,

def of integral

$$\int_\gamma F'(z)\,dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} F'(\gamma(t))\gamma'(t)\,dt$$

$$= \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} \left[u'(t) + i v'(t)\right] dt$$

$$= \sum_{i=0}^{n-1} \left(\int_{a_i}^{a_{i+1}} u'(t)\,dt \right) + i \left(\int_{a_i}^{a_{i+1}} v'(t)\,dt \right)$$

$$= \sum_{i=0}^{n-1} \left\{ \left(u(a_{i+1}) - u(a_i)\right) + i\left(v(a_{i+1}) - v(a_i)\right) \right\}$$

Calc II FTOC

$$= \left(u(a_n) - u(a_{n-1})\right) + i\left(v(a_n) - v(a_{n-1})\right) \quad \leftarrow i = n-1$$

$$+ \left(u(a_{n-1}) - u(a_{n-2})\right) + i\left(v(a_{n-1}) - v(a_{n-2})\right) \quad \leftarrow i = n-2$$

$$\cdots$$

$$+ \left(u(a_1) - u(a_0)\right) + i\left(v(a_1) - v(a_0)\right) \quad \leftarrow i = 0$$

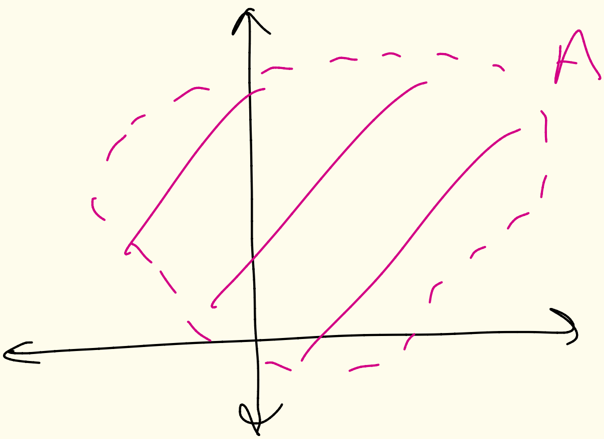
lots of cancelling

$$= \left(u(a_n) + i v(a_n)\right) - \left(u(a_0) + i v(a_0)\right) =$$

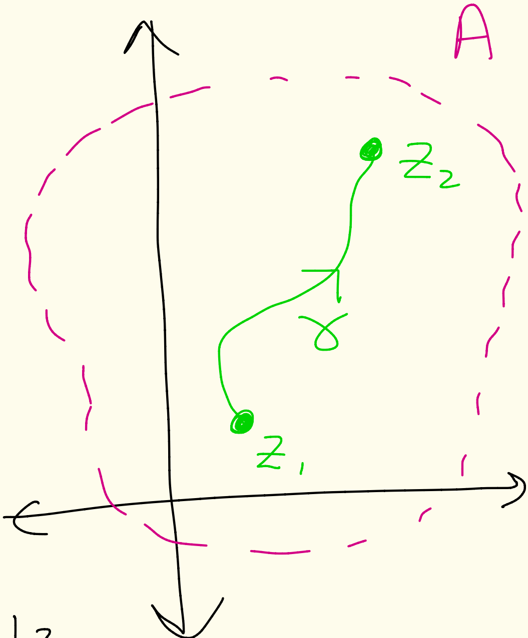
$$= F(\gamma(b)) - F(\gamma(a))$$

$\blacksquare$

<u>Theorem:</u> Suppose that A is a region (open and path-connected) and that $f: A \to \mathbb{C}$ is analytic on A and $f'(z) = 0$ for all $z \in A$. Then $f(z) = c$ for all $z \in A$ for some constant $c \in \mathbb{C}$.



<u>Proof:</u> Let z_1 and z_2 be any two points in A. We will show that $f(z_1) = f(z_2)$. Thus, f is constant in A.



Since A is path-connected there must exist a piecewise-smooth curve γ from z_1 to z_2.

Then,

$$0 = \int_\gamma 0\,dz \underset{\boxed{f'(z)=0 \text{ in } A}}{=} \int_\gamma f'(z)\,dz$$

$$\overset{\boxed{\text{FTOC}}}{=} f(z_2) - f(z_1).$$

So, $f(z_1) = f(z_2)$.

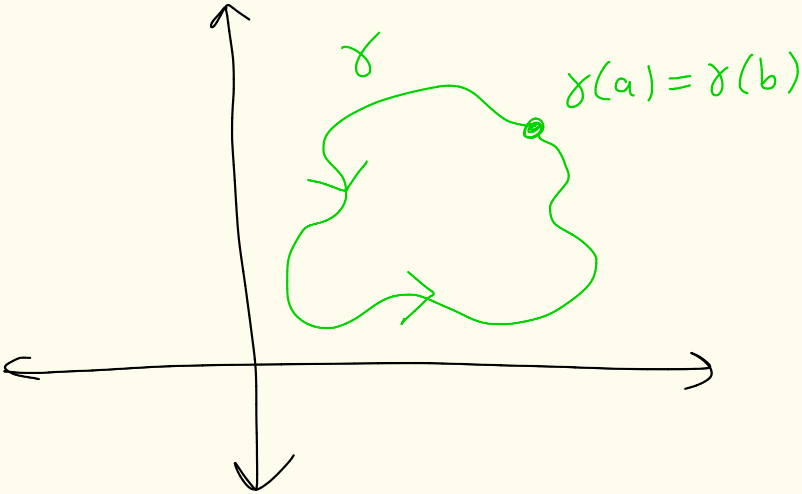
$\blacksquare$

<u>Def:</u> A curve

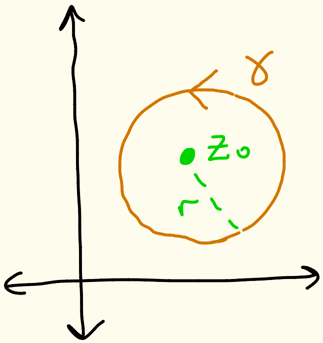
$\gamma : [a,b] \to \mathbb{C}$

is called <u>closed</u>

if $\gamma(a) = \gamma(b)$.

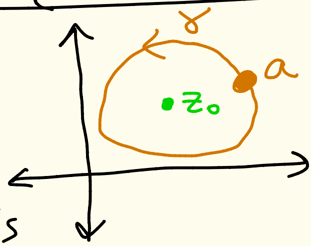


Ex: Let γ be a parameterization of the circle of radius r centered at z_0 oriented counterclockwise.



Then,

$$\int_\gamma (z-z_0)^n dz = \begin{cases} 0, & \text{if } n \neq -1 \\ 2\pi i, & \text{if } n = -1 \end{cases}$$



case 1: Suppose $n \geq 0$.

Let $F(z) = \frac{1}{n+1}(z-z_0)^{n+1}$. F is analytic on all of $\mathbb{C}$ and $F'(z) = (z-z_0)^n$

So by FTOC, if we pick some point a on γ then $\int_\gamma (z-z_0)^n dz = F(a) - F(a) = 0$.

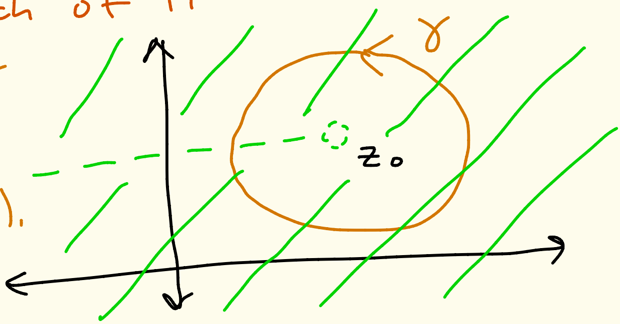
case 2: Suppose $n < -1$. Use the same F as above, but now F is analytic on $\mathbb{C} - \{z_0\}$, which contains γ. So again we can still use FTOC and get

$\int_\gamma (z-z_0)^n dz = F(a) - F(a) = 0$, where a is some point on γ.

Case 3: In this case, $\dfrac{1}{z-z_0}$ has no antiderivative on an open set containing γ.

Because the function $F(z)=\log(z-z_0)$ has $F'(z)=\dfrac{1}{z-z_0}$ but you need to do a branch of it which will hit γ.

Green part is domain of branch of $\log(z-z_0)$.



So $\log(z-z_0)$ isn't analytic on an open set containing γ

So we can't use FTOC. Let's just calculate it using the def of integral.

parametrize the curve:

$$\gamma(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi.$$

If $z_0 = x_0 + iy_0$ then

$$\gamma(t) = (x_0 + iy_0) + r[\cos(t) + i\sin(t)]$$
$$= (x_0 + r\cos(t)) + i(y_0 + r\sin(t))$$

$$\gamma'(t) = -r\sin(t) + ir\cos(t)$$
$$= ir[\cos(t) + i\sin(t)]$$
$$= ire^{it}$$

So,

$$\int_\gamma \frac{dz}{z - z_0} = \int_0^{2\pi} \underbrace{\frac{1}{(z_0 + re^{it} - z_0)}}_{\left(\frac{1}{\gamma(t) - z_0}\right)} \cdot \underbrace{rie^{it}}_{\gamma'(t)} \, dt$$

$$= \int_0^{2\pi} i \, dt = i \int_0^{2\pi} dt = it \Big|_0^{2\pi} = 2\pi i$$

$\boxed{\text{Example}}$