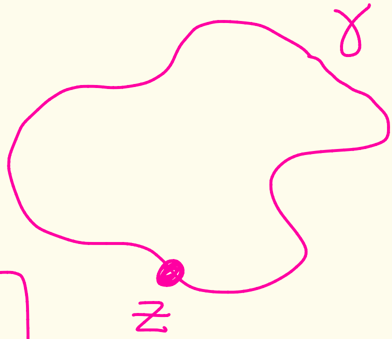
TOPIC 8 – Rouché's Theorem

Rouché's Thm

Let γ be a simple, closed, piecewise-smooth curve.

Suppose f and h are analytic inside γ and on γ.



Suppose

$$|h(z)| < |f(z)|$$

for all z on γ.

Then, f and $f+h$ have the same number of zeros inside of γ (counting multiplicities).

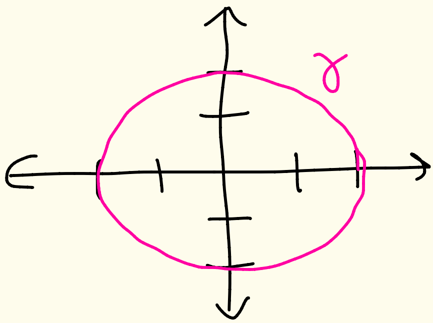
Proof: Later

Ex: Show that

$$p(z) = z^5 + 3z + 1$$

has 5 zeros (counting multiplicity) inside the curve $|z| = 2$.

Let γ be the curve $|z| = 2$.



Let $f(z) = z^5$

and $h(z) = 3z + 1$.

If z is on γ,

ie $|z| = 2$, then

$$|h(z)| = |3z+1| \leq |3z| + |1|$$
$$= |3||z| + 1$$
$$= 3 \cdot 2 + 1 = 7$$

and

$$|f(z)| = |z^5| = |z|^5 = 2^5 = 32.$$

Thus, if z is on γ, then $|h(z)| \leq 7 < 32 = |f(z)|$

Thus, by Rouché's theorem

$$f(z) = z^5$$

and

$$p(z) = f(z) + h(z) = z^5 + 3z + 1$$

have the same number of zeros inside γ.

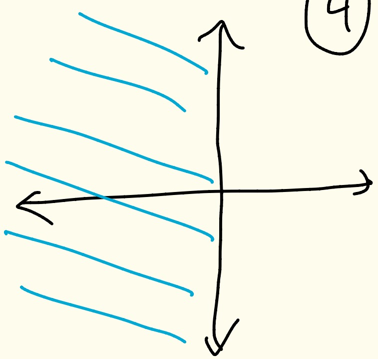
Since f has 5 zeroes inside γ (counting multiplicity),

so does $p(z)$. $\blacksquare$

<u>Ex:</u> Show that

$$p(z) = z + 3 + 2e^{z}$$

has one zero in the left half-plane.



Let $f(z) = z+3$

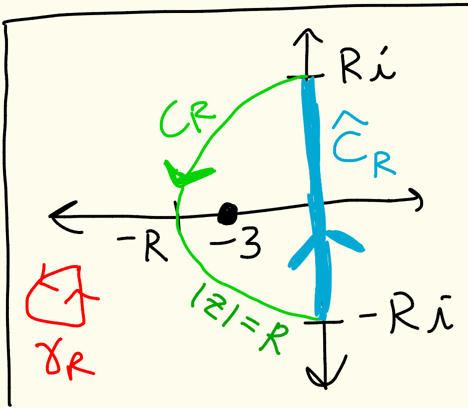
and $h(z) = 2e^{z}$

So that $p(z) = f(z) + h(z)$

and f has 1 zero in the left half plane at $z=-3$.

Let C_R and $\widehat{C}_R$ be as in the picture and let $\gamma_R = \widehat{C}_R + C_R$

We only look at $R > 3$.



Let z be on $\widehat{C}_R$. ⑤

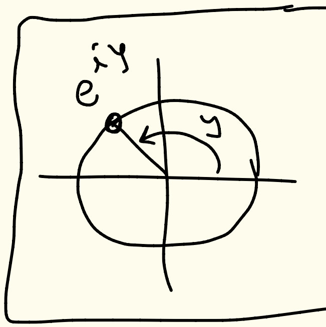
Then, $z = 0 + iy$ where $-R \leq y \leq R$.

So,

$$|h(z)| = |2e^z| = |2e^{0+iy}| = 2 \cdot \underbrace{|e^{iy}|}_{1}$$

$$= 2$$

and



$$|f(z)| = |z+3|$$
$$= |iy+3|$$
$$= |3+iy|$$
$$= \sqrt{3^2+y^2}$$
$$\geq \sqrt{3^2} = 3$$

So when z is on $\widehat{C}_R$ we have

$$|h(z)| = 2 < 3 \leq |f(z)|.$$

Let z be on C_R ⑥

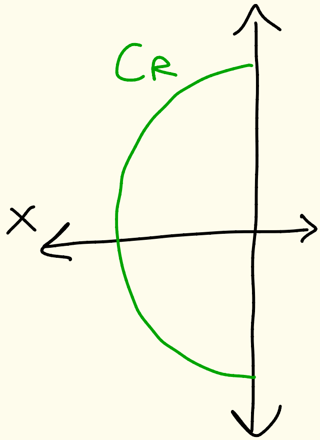
Then $|z| = R > 3$.

In this case, when $z = x+iy$ we have

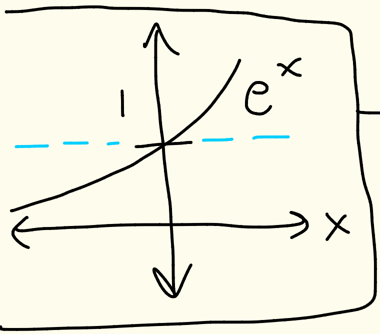
$$|h(z)| = |2e^z|$$

$$= 2|e^{x+iy}|$$

$$= 2|e^x|\underbrace{|e^{iy}|}_{1}$$



$$= 2e^x \leq 2$$



$\boxed{x \leq 0}$

$\boxed{e^x \leq 1}$

And,

$$|f(z)| = |z+3| \geq \big||z|-3\big|$$

$$= |R-3| = R-3$$

$\boxed{R>3}$

So if $R > 5$ and z is on C_R we have

$$|h(z)| \leq 2 < R - 3 \leq |f(z)|.$$

$R > 5$

Thus, if $R > 5$ then

$$|h(z)| < |f(z)|$$

for all z on $\gamma_R = C_R + \hat{C}_R$

By Rouché's theorem,

$$p(z) = f(z) + h(z) = z + 3 + 2e^z$$

has the same number of zeros inside γ_R $(R > 5)$ as $f(z) = z+3$ does. So, $p(z)$ has one zero inside γ_R for any $R > 5$. Letting $R \to \infty$ we get that $p(z)$ has one zero in the left-half plane.

Now we prove Rouché's thm.

Note to Tony: Go back and prove the identity theorem first

Argument principle / Rouche Thm

$\boxed{\text{Setup } (*)}$

γ is a simple, closed, piecewise smooth curve oriented counterclockwise. $f(z)$ is analytic on and inside γ, except for (possibly) some finite poles inside (not on) γ and some zeros inside (not on) γ

Let $p_1, \ldots, p_m$ be the poles of f inside γ.

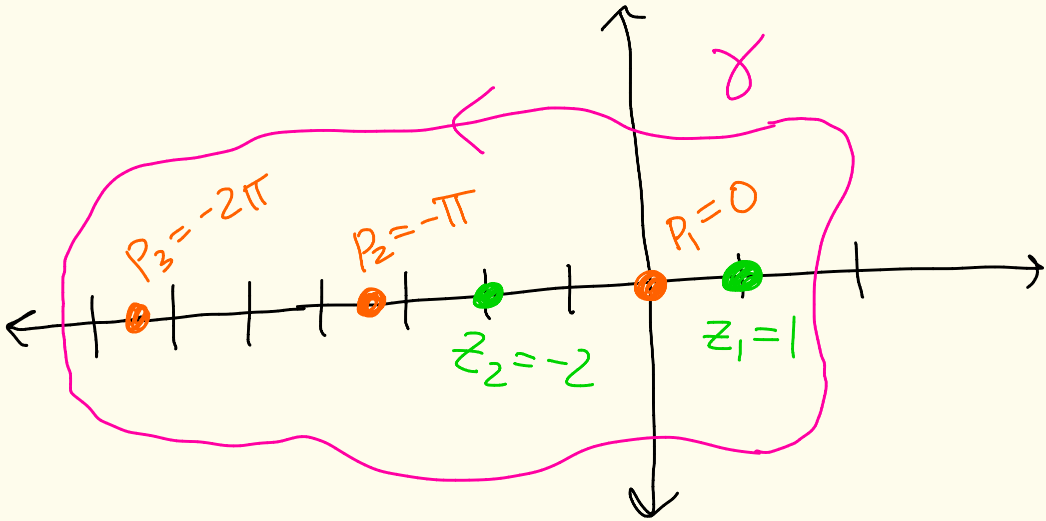
Let $z_1, \ldots, z_n$ be the zeros of f inside γ

Write $\text{mult}(z_k) =$ multiplicity of the zero z_k. Write $\text{mult}(p_k) =$ order of pole p_k

Ex:

$$f(z) = \frac{(z-1)^3(z+2)}{\sin(z)}$$

$2\pi \approx 6.28$



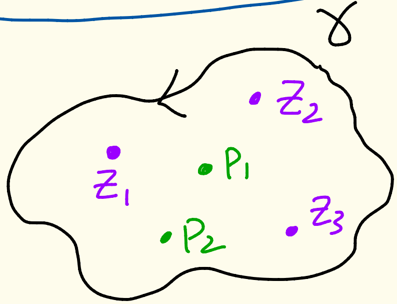
$\text{mult}(z_1) = 3$

$\text{mult}(z_2) = 1$

$\left.\begin{array}{l} \text{mult}(p_1) = 1 \\ \text{mult}(p_2) = 1 \\ \text{mult}(p_3) = 1 \end{array}\right\}$

$\boxed{\text{$0, -\pi, -2\pi$ are zeros of order 1 of $\sin(z)$}}$

Theorem: Suppose we have $\boxed{\text{setup } (*)}$ for a function f and curve γ.



picture when
$m = 2$
$n = 3$

Then,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\left[\sum_{k=1}^{n} \mathrm{mult}(z_k) - \sum_{k=1}^{m} \mathrm{mult}(p_k)\right]$$

proof: We need to understand the poles and residues of $\frac{f'(z)}{f(z)}$ inside γ.

The only possible poles would be at the z_k and p_k.

$$f(z) = \frac{1}{z}$$

$$\frac{f'}{f} = \frac{-\frac{1}{z^2}}{\frac{1}{z}} = -\frac{1}{z}$$

Suppose f has a zero of order ℓ at some z_k inside γ.
By setup (*) there are only a finite number of zeros of f inside γ.
Thus, z_k is an isolated zero.
So, $\boxed{f(z) = (z - z_k)^{\ell} \varphi(z)}$ ← this is for z near z_k

where $\ell = \text{mult}(z_k)$ and φ is analytic at z_k and $\varphi(z_k) \neq 0$

Then, near z_k we have

$$\frac{f'(z)}{f(z)} = \frac{\ell(z-z_k)^{\ell-1}\varphi(z) + (z-z_k)^{\ell}\varphi'(z)}{(z-z_k)^{\ell}\varphi(z)}$$

$$= \frac{\ell}{z-z_k} + \frac{\varphi'(z)}{\varphi(z)}$$

Note that because $\varphi(z_k) \neq 0$ and φ and φ' are analytic at z_k, we know φ'/φ is analytic at z_k.

Thus, $\frac{f'}{f}$ has a simple pole at each z_k and

$$\operatorname{Res}\left(\frac{f'}{f}; z_k\right) = \operatorname{mult}(z_k).$$

Now suppose p_k is a pole inside of γ of order t.

It's an isolated singularity

So we can write

$$f(z) = \frac{\beta(z)}{(z-p_k)^t}$$

this is for z near p_k

Where β is analytic at p_k and $\beta(p_k) \neq 0$.

Then for z near p_k we have

$$\frac{f'(z)}{f(z)} = \frac{-t(z-p_k)^{-t-1}\beta(z) + (z-p_k)^{-t}\beta'(z)}{(z-p_k)^{-t}\beta(z)}$$

$$= \frac{-t}{(z-p_k)} + \frac{\beta'(z)}{\beta(z)}$$

Again $\frac{\beta'}{\beta}$ is analytic at p_k $[\beta(p_k) \neq 0]$.

Thus, $\frac{f'}{f}$ has a simple pole at p_k and $\text{Res}\left(\frac{f'}{f}; p_k\right) = -\text{mult}(p_k)$

By the Residue theorem

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i \sum \left(\begin{array}{c}\text{residues of } \frac{f'}{f} \\ \text{at each of its} \\ \text{poles}\end{array}\right)$$

$$= 2\pi i \left[\sum_{k=1}^{n} \text{mult}(z_k) - \sum_{k=1}^{m} \text{mult}(p_k)\right]$$

$\blacksquare$

Notation: Suppose we have setup (*)

$$Z_{f,\gamma} = \sum_{k=1}^{n} \text{mult}(z_k)$$

$$P_{f,\gamma} = \sum_{k=1}^{m} \text{mult}(p_k)$$

So the previous theorem says

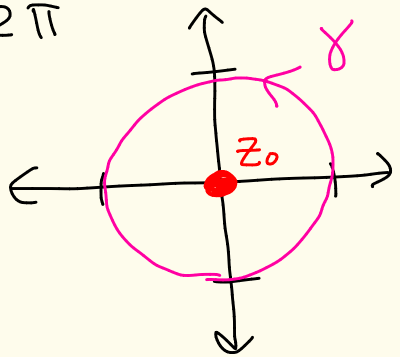
$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\left[Z_{f,\gamma} - P_{f,\gamma}\right]$$

Def: Let γ be a piecewise smooth closed curve. The *winding number* (or *index*) of γ about z_0 is defined to be

$$\mathrm{Ind}(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz$$

Ex: $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$

$z_0 = 0$



$$\mathrm{Ind}(\gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - 0}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{e^{it} - 0} \cdot i e^{it}\, dt$$

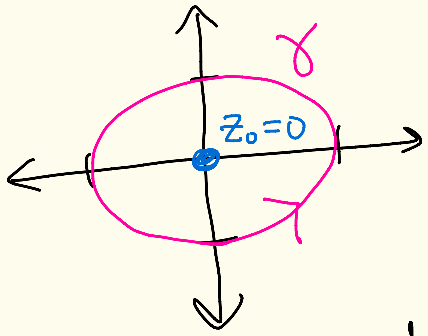
$\gamma'(t) = i e^{it}$

$$= \frac{1}{2\pi i} \int_0^{2\pi} i\, dt = \frac{1}{2\pi} \overbrace{\int_0^{2\pi} 1\, dt}^{2\pi} = 1$$

Ex: $\gamma(t) = e^{10it}$, $0 \le t \le 2\pi$

$z_0 = 0$

$\gamma'(t) = 10ie^{10it}$



γ wraps around 0, 10 times

$$\text{Ind}(\gamma; 0) =$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{10it} - 0} \cdot 10ie^{10it}\,dt$$

$$= \frac{10i}{2\pi i}\int_0^{2\pi} 1\,dt = \frac{5}{\pi}\left[2\pi\right] = 10$$

Note: If the curve goes clockwise it makes the answer negative.

Ex: Suppose γ is a simple, piecewise smooth closed curve. Suppose z_0 is not on γ.

Let $\boxed{f(z) = z - z_0}$

Then $f'(z) = 1$.

Then we have setup (*) so

$$\text{Ind}(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, dz$$

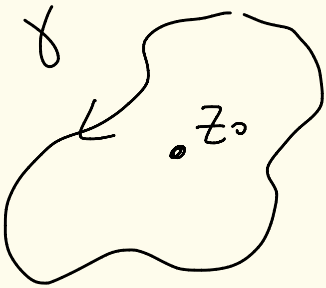
$$= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz$$

$$= \frac{1}{2\pi i} \left[2\pi i \cdot \left\{ Z_{f,\gamma} - \underbrace{P_{f,\gamma}}_{0} \right\} \right] = Z_{f,\gamma}$$

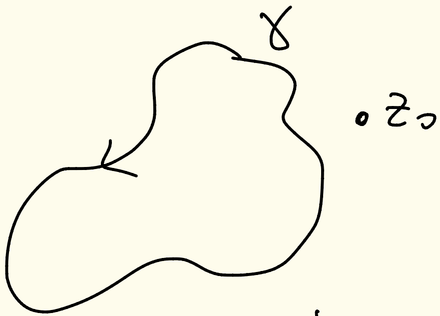
And $Z_{f,\gamma} = \begin{cases} 1, & \text{if } z_0 \text{ is inside } \gamma \\ 0, & \text{if } z_0 \text{ is not inside } \gamma \end{cases}$

So, in this case

$$\mathrm{Ind}(\gamma; z_0) = \begin{cases} 1 & \text{if } z_0 \text{ is inside } \gamma \\ 0 & \text{if } z_0 \text{ is not inside } \gamma \end{cases}$$



z_0 is inside γ



z_0 is not inside γ

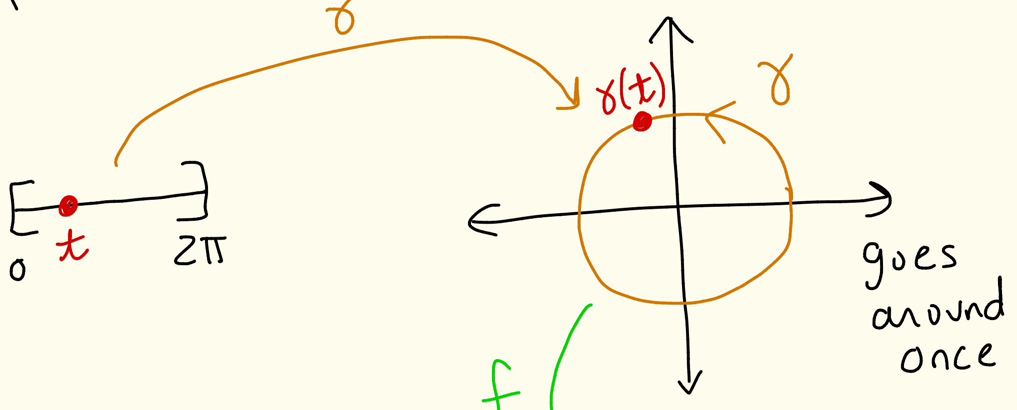
<u>Ex:</u> Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$

Let $f(z) = z^2$

Describe the curve $f \circ \gamma$

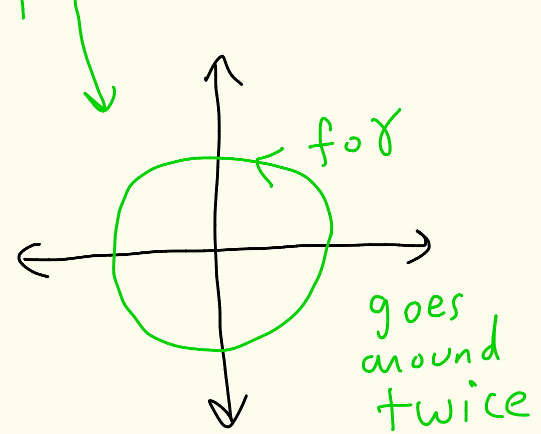
$$(f \circ \gamma)(t) = f(\gamma(t)) = f(e^{it}) = e^{2it}$$

$$0 \le t \le 2\pi$$



$\text{Ind}(\gamma; 0) = 1$

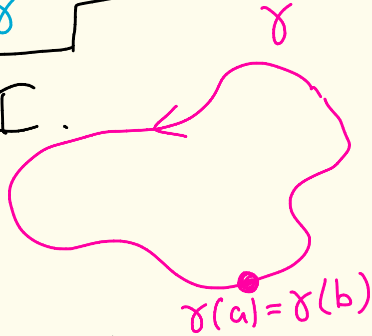
$\text{Ind}(f \circ \gamma; 0) = 2$



Theorem: Suppose we have setup (*) for a function f and curve γ. Then,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i \underbrace{\mathrm{Ind}(f\circ\gamma;0)}_{\frac{1}{2\pi i}\int_{f\circ\gamma}\frac{1}{z-0}\,dz}$$

proof: Let $\gamma:[a,b]\to\mathbb{C}$.



We have that

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\cdot\gamma'(t)\,dt$$

$$= \int_{f\circ\gamma}\frac{1}{z}\,dz = \int_{f\circ\gamma}\frac{1}{z-0}\,dw$$

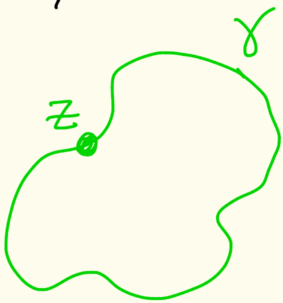
$$(f\circ\gamma)'(t) = f'(\gamma(t))\cdot\gamma'(t)$$

$$= 2\pi i\,\mathrm{Ind}(f\circ\gamma;0)$$

Note: $\int_{\gamma} \frac{1}{z}\,dz$ is okay because f has no zeros on γ by setup (*). ∎

Rouché's Theorem: Let γ be a simple, closed, piecewise smooth curve.
Let f and h be analytic on and inside γ, except for (possibly) some finite poles inside (not on) γ.

Also assume
$$|h(z)| < |f(z)|$$
for all z on γ.



Then, $Z_{f,\gamma} - P_{f,\gamma} = Z_{f+h,\gamma} - P_{f+h,\gamma}$

<u>Proof:</u> We are going to use the previous theorems and apply them to the functions f, $f+h$, $\frac{f+h}{f}$.

So we need to show that these functions satisfy setup $(*)$, ie they have no poles or zeros on γ.

<u>Zeros:</u> The fact that $0 \le |h(z)| < |f(z)|$ for all z on γ implies that f has no zeros on γ.

Why is $f+h$ not zero on γ?

Suppose $(f+h)(z) = 0$ for some z on γ.

Then $f(z) + h(z) = 0$.

So, $f(z) = -h(z)$.

And, $\underline{|f(z)| = |h(z)|}$ which can't happen.

This also shows that $\frac{f+h}{f}$ can't be zero on γ since $f+h$ has no zeros on γ.

Poles: Since f and h have no poles on γ, we know $f+h$ has no poles on γ.

[f and h are analytic on γ, so $f+h$ is analytic on γ]

Since f has no zeros on γ, $\frac{f+h}{f}$ will have no poles on γ.

So, f, $f+h$, $\frac{f+h}{f}$ all satisfy the setup $(*)$ conditions.

Thus,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \mathrm{Ind}(f\circ\gamma;0) = Z_{f,\gamma} - P_{f,\gamma}$$

$$\frac{1}{2\pi i}\int_\gamma \frac{(f+h)'(z)}{(f+h)(z)}\,dz = \mathrm{Ind}((f+h)\circ\gamma;0) = Z_{f+h,\gamma} - P_{f+h,\gamma}$$

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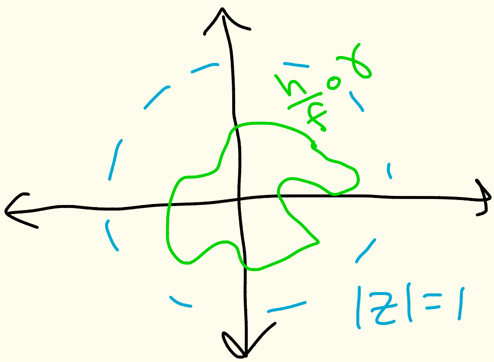
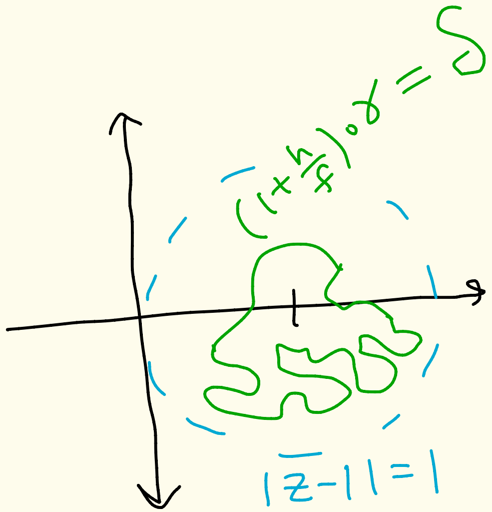
By assumption, $\left|\frac{h(z)}{f(z)}\right| < 1$ for all z on γ.

Thus, $\left(\frac{h}{f}\right)\circ\gamma$ is inside the unit circle

Thus, $1+\frac{h}{f} = \frac{f+h}{f}$ maps γ to the inside of the unit disc centered at 1.

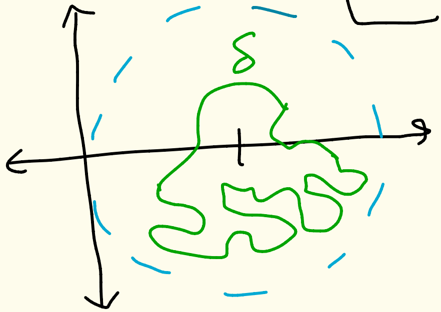




Let δ be the image of γ under $\frac{f+h}{f}$. That is,

$$\delta = \left(\frac{f+h}{f}\right)\circ\gamma = \left(1+\frac{h}{f}\right)\circ\gamma$$

Thus,



$$\mathrm{Ind}\Big(\underbrace{\left(\frac{f+h}{f}\right)\circ\gamma}_{\delta},\ 0\Big)$$

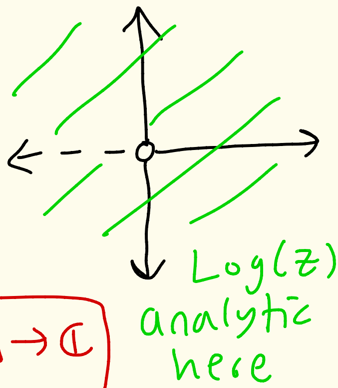
$$= \frac{1}{2\pi i}\int_{\delta}\frac{1}{z-0}\,dz$$

$$= \frac{1}{2\pi i}\int_{\delta}\frac{1}{z}\,dz$$

$$= \frac{1}{2\pi i}\Big[\mathrm{Log}(\delta(b)) - \mathrm{Log}(\delta(a))\Big] = 0$$

FTOC
Log(z) is the principal branch

Suppose $\gamma : [a,b] \to \mathbb{C}$



$$\delta(b) = \left(\frac{f+h}{f}\right)(\gamma(b)) = \left(\frac{f+h}{f}\right)(\gamma(a)) = \delta(a)$$

Let $g = \dfrac{f+h}{f}$.

So, $\operatorname{Ind}(\underbrace{g \circ \gamma}_{\delta} ; 0) = 0$

So, $$\int_\gamma \frac{g'(z)}{g(z)}\,dz = 2\pi i \operatorname{Ind}(g\circ\gamma ; 0)$$
$$= 0$$

$\boxed{\dfrac{f+h}{f} \text{ satisfies setup } (*)}$

Note that on γ we have

$$\frac{g'}{g} = \frac{\left(\frac{f+h}{f}\right)'}{\left(\frac{f+h}{f}\right)} = \frac{(f+h)'f - f'(f+h)}{f^2} \cdot \frac{f}{f+h}$$

$$= \frac{(f+h)'f - f'(f+h)}{f(f+h)} = \frac{(f+h)'}{f+h} - \frac{f'}{f}$$

Thus,

$$0 = \int_\gamma \frac{g'(z)}{g(z)}\,dz$$

$$= \int_\gamma \frac{(f+h)'(z)}{(f+h)(z)}\,dz - \int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

So,

$$Z_{f,\gamma} - P_{f,\gamma} = \int_\gamma \frac{f'(z)}{f(z)}\,dz$$

$$= \int_\gamma \frac{(f+h)'(z)}{(f+h)(z)}\,dz = Z_{f+h,\gamma} - P_{f+h,\gamma}$$

$\blacksquare$

<u>Corollary (Rouché's thm)</u> Under the same conditions as the above theorem if f and h have no poles inside γ then $Z_{f,\gamma} = Z_{f+h,\gamma}$