Comprehensive Examination – Topology

Fall 2001

Do five problems, including the first one. Each problem is worth 20 points. The set of positive integers is denoted by N, the set of rationals by Q, and the set of real numbers by R. The notation A^c means the complement of the set A with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Explain carefully 5 of the following concepts:

(a) Connected component of a topological space.

(b) Uniformly continuous function between two metric spaces.

(c) Normal space.

(d) Product topology.

(e) Compact topological space.

(f) Basis for a topology.

2. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a continuous function. Define $\rho : X \times X \to R$ by $\rho(x_1, x_2) = d_X(x_1, x_2) + d_Y(f(x_1), f(x_2))$. Prove that

(a) ρ is a metric on X.

(b) The metrics ρ and d_X are equivalent.

3. Let A, B be subsets of a topological space X. Prove or disprove each of the following statements:

(a) $\text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B)$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\overline{A} \cap B = \emptyset$ implies that $A \cap \overline{B} = \emptyset$.

4. Let X and Y be topological spaces. Suppose that X is normal and $f : X \to Y$ is continuous, open, and onto. Prove that Y is normal.

5. Let X be a topological space and A be a subset of X.

(a) Define $\text{bd}(A)$, the boundary of A.

(b) Use the definition in (a) to prove that: A is open if and only if $A \cap \text{bd}(A) = \emptyset$.

(c) Use the definition in (a) to prove that: A is closed if and only if $\text{bd}(A) \subseteq A$.

6. Let X, Y be metric spaces, X being compact. Prove that a continuous mapping $f : X \to Y$ is uniformly continuous.

7. Let $f : X \to Y$ be a continuous mapping.

(a) Prove or disprove: If $C \subseteq Y$ is connected then $f^{-1}(C)$ is connected.

(b) Prove or disprove: If $C \subseteq X$ is connected then $f(C)$ is connected.

(c) State a theorem from Calculus which is a particular case of (a) or (b).