# Comprehensive Examination – Topology

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Do any five of the problems that follow. Each problem is worth 20 points. The set of positive integers is denotes by $N$, the set of rationals by $Q$, and the set of real numbers by $R$. The notation $A^c$ means the complement of the set $A$ with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

**1.** Prove that each metric space $(X, d)$ is normal.

**2.** (a) Give an example of a connected topological space that is not locally connected.

(b) Give an example of a topological space that is not separable. Justify your answer.

**3.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $f : X \to Y$ be continuous.

(a) Assuming that $Y$ is Hausdorff, prove that the graph of $f$, $\Gamma(f) = \{(x, y) : x \in X, y = f(x)\}$ is a closed subset of $X \times Y$ equipped with the product topology.

(b) Prove that $\Gamma(f)$ equipped with the relative topology is homeomorphic to $X$.

**4.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $f : X \to Y$. Prove that $f$ is continuous if and only if for each subset $A$ of $X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$. Here $\overline{E}$ denotes the closure of $E$.

**5.** Let $(X, \tau)$ be a compact Hausdorff space.

(a) Let $F$ be a nonempty closed subset of $X$. Prove that $F$ equipped with the relative topology is compact.

(b) Let $f : X \to X$ be continuous, one-to-one, and onto. Prove that $f$ is a homeomorphism.

**6.** (a) What is meant by diam$(A)$, the diameter of a nonempty subset $A$ of a metric space $(X, d)$?

(b) Let $< A_n >$ be a sequence of nonempty closed sets in a complete metric space $(X, d)$ such that for any $n$ $A_{n+1} \subseteq A_n$ and $\lim_{n\to\infty} \text{diam}(A) = 0$. Prove that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

**7.** Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces.

(a) Suppose that $A$ is closed subset of $X$ and $B$ is closed subset of $Y$. Prove or produce a counterexample: $A \times B$ is a closed subset of $X \times Y$ equipped with the product topology.

(b) Suppose $E$ is a closed subset of $X \times Y$ equipped with the product topology. Prove or produce a counterexample: $\pi_X(E) = \{x \in X : \exists\, y \in Y \text{ with } (x, y) \in E\}$.