Spring 2006 Topology Comprehensive Exam

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Do any five (5) of the problems that follow. Each is worth 20 points.
Let $\mathbb{R}$ denote the set of real numbers.

1. Let (X, τ_A) be a topological space. Equip $X \times X$ with the product topology.

(a) Prove that X is Hausdorff iff the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

(b) Prove that if A and B are closed in X, then $A \times B$ is closed in $X \times X$.

2. Determine whether or not each pair A, B of topological spaces is homeomorphic. In each case, prove that your answer is correct.

(a) Let $A = (0, 1)$ be the open unit interval in the real line, and let $B = \mathbb{R}$ (both with the usual topology).

(b) Let $A = [0, 1]$ be the closed unit interval in the real line, and let $B = \mathbb{R}$ (both with the usual topology).

(c) Let $A = B = \mathbb{R}$, where A has the usual topology and B has the cofinite topology (that is, if $C \subset B$ is a closed set, then either C is finite or $C = \mathbb{R}$).

(d) Let $A = \mathbb{R}$, and let B be the union of the x-axis and the y-axis in $\mathbb{R}^2$.

3. Let $\mathbb{Z}$ be the set of all integers. For all $a, b \in \mathbb{Z}$, let $B_{a,b} = \{an + b \mid n \in \mathbb{Z}\}$. (For example, $B_{2,1}$ is the set of odd numbers.)

(a) Show that $\{B_{a,b} \mid a, b \in \mathbb{Z}\}$ is a basis for a topology on $\mathbb{Z}$.

(b) Show that every nonempty open subset in this topology is infinite.

(c) Show that $B_{a,b}$ is closed in this topology for all $a, b \in \mathbb{Z}$.

(d) Prove that there are infinitely many prime numbers. (Hint: Consider $\cup_{p \in S} B_{p,0}$, where S is the set of all prime numbers. You may assume without proof that every positive integer greater than 1 has a prime factorization.)

4. Let (X, d) be a compact metric space. Prove that this space is second countable.

5. Let (X, τ) be a compact Hausdorff space and let τ_1 and τ_2 be topologies on X that satisfy $\tau_1 \subset \tau \subset \tau_2$ and $\tau_1 \neq \tau$ and $\tau \neq \tau_2$. Prove the following:

(a) (X, τ_1) is not a Hausdorff space.

(b) (X, τ_2) is not compact.

(Hint: consider the identity map $\iota : X \to X$.)

6. Let X and Y be topological spaces. Let $Z = X \times Y$, and endow Z with the product topology. Prove or disprove: Z is connected iff X and Y are each connected.

7. Prove (a) and (b) below.

(a) Every subspace of a completely regular space is completely regular.

(b) Every closed subspace of a normal space is normal.