Topology Comprehensive Exam Fall 2007

Beer and Krebs

Do 5 of the 7 problems below. Each is worth 20 points

1. A topological space X is *reducible* iff there exist two nonempty **proper** closed subsets A, B of X such that $A \bigcup B = X$. (Note that A and B do not have to be disjoint.) The topological space X is *irreducible* iff X is not reducible

   a. Prove that if X is an irreducible topological space, then X is connected.

   b. Prove that if X is an irreducible topological space and X contains at least two points then X is not Hausdorff.

2. Let $\mathbb{R}$ be the set of real numbers with the usual metric. Let Y be a metric space.

   (a) Prove that if $f : \mathbb{R} \to Y$ is continuous and S is closed and bounded in $\mathbb{R}$, then $f(S)$ is closed and bounded in Y.

   (b) Give an example to show that the following statement is not true in general: if X is a metric space and if $g: X \to \mathbb{R}$ is continuous and T is closed and bounded in X, then $g(T)$ is closed and bounded in $\mathbb{R}$. (Hint: an example can be constructed when X is $(0,1)$ as a subspace of $\mathbb{R}$ or X is an infinite set equipped with the discrete metric. Whatever example you choose to present, argue that it has the asserted properties).

3. By a *linear order* $\leq$ on a set X, we mean a relation that is reflexive, antisymmetric and transitive and such that each two elements of X are comparable. Write $x < w$ in X if $x \leq w$ but $x \neq w$. The *order topology* τ on a linearly ordered set X is the topology generated by all "rays" of the form $\{x \in X : x < w\}$ and $\{x \in X : x > w\}$ where $w \in X$.

   (a) Prove that the order topology is Hausdorff.

   (b) Suppose the order topology is compact. Show there exists some x_1 in X such that for all x in X, $x_1 \leq x$.

4. Let $f : X \to Y$ be a continuous function of topological spaces, and consider the set

$$\Gamma = \{(x, f(x)) : x \in X\} \subset X \times Y$$

as a subspace of the product space $X \times Y$. Prove that Γ is homeomorphic to X.

5. Let X be a topological space and let $\{C_k\}_{k\in Z} \subset X$ denote a collection of connected subsets indexed by the integers $\mathbb{Z}$. Suppose that

$$\forall k, C_k \cap C_{k+1} \neq \phi.$$

Prove that $\bigcup_{k\in Z} C_k$ is connected. (Note : the index set is **not** the positive integers!)

6. Prove that each metric space (X,d) is normal.

7. Precisely explain each of the following results or concepts.

(a) *Base* β for a topology τ on a set X;

(b) The *product topology* for a product of topological spaces $\{(X_i, \tau_i) : i \in I\}$;

(c) Urysohn' lemma;

(d) *Regular* topological space;

(e) *Complete* metric space (X,d);

(f) A *path* from point x to point w in a topological space X;

(g) *Boundary point* p of a set A in a topological space X.