Fall 2009 Topology Comprehensive Exam

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Do any 5 of the 7 problems below; they are worth 20 points each

1. Let $\mathbb{R}$ denote the set of real numbers. Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$. (In other words, X equals the plane minus the origin.) Topologize X by taking the open sets to be $\emptyset$, X, and every set of the form $X \setminus (\ell_1 \cup \cdots \cup \ell_n)$ for some positive integer n and lines $\ell_1, \ldots, \ell_n$ through the origin.

(a) Prove that X is connected.

(b) Prove that X is not Hausdorff.

2. Give an example of a bijective continuous function which is not a homeomorphism.

3. Let X and Y be topological spaces such that X is compact. Suppose that f is a continuous function from X to Y such that f is surjective. Let (A_j) be a decreasing sequence of nonempty closed sets in Y. (In other words, for all j we have that A_j is closed in Y and $A_j \neq \emptyset$ and $A_{j+1} \subseteq A_j$.) Prove that $\bigcap_{j=1}^{\infty} f^{-1}(A_j)$ is nonempty.

4. (a) Explain how you know that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

(b) Produce a homeomorphism from $(2, 8)$ to $(-1, 1)$.

(c) Produce a homeomorphism from $(2, 8)$ to $(-\infty, \infty)$.

Note: in parts (b) and (c) you need only exhibit the function; you need not prove that it is a homeomorphism.

5. (a) Prove that a separable metric space $\langle X, d\rangle$ is second countable.

(b) Give an example of a metric space $\langle X, d\rangle$ that is not separable.

6. Let $\{Y_i : i \in I\}$ be a collection of topological spaces. Equip $\prod_{i\in I} Y_i$ with the product topology and for each $i \in I$, let π_i be the projection map from the product onto Y_i. Prove that if X is a topological space, then a function $f : X \to \prod_{i\in I} Y_i$ is continuous if and only if $\forall i \in I,\ \pi_i \circ f$ is continuous.

7. Prove that each metric space $\langle X, d\rangle$ is normal.