TOPOLOGY COMPREHENSIVE EXAMINATION FALL 2013

(Akis*, Beer, Krebs)

DO 5 OUT OF 7 (Indicate clearly which 5 problems you submit for evaluation).

1. (a) Let X be a set. Prove that the intersection of any nonempty collection of topologies on X is a topology.

(b) Give an example to show that the union of two topologies need not be a topology.

2. Let X be a normal Hausdorff space. Let R denote the set of real numbers, endowed with the usual topology. We say that a subset U of X is a cozero set if there exists a continuous function $f : X \to R$ such that $U = \{x \in X : f(x) \neq 0\}$. Prove that the collection of all cozero sets forms a basis for the given topology of X.

3. (a) Let $\{K_1, K_2, \cdots, K_n\}$ be a family of nonempty compact subsets of a topological space X. Prove $\bigcup_{j=1}^{n} K_j$ is compact.

(b) Let F be a nonempty closed subset of a compact set $K \subseteq X$. Prove that F is compact.

4. Consider the Euclidean plane R^2 equipped with the usual topology. Prove, giving complete justification of all claims, that

(a) $\{(x, y) : x^2 + y^2 > 1\}$ is a connected subset of R^2;

(b) $\{(x, y) : x^2 + y^2 \neq 1\}$ fails to be a connected subset of R^2.

5. Let $\langle x_n \rangle$ be a Cauchy sequence in a metric space (X,d).

(a) Prove that $E = \{x_n : n \in \mathbf{N}\}$ is a bounded set, i.e., that E is contained in a ball.

(b) Suppose $\langle w_n \rangle$ is another sequence in the space where $\lim_{n\to\infty} d(x_n, w_n) = 0$. Prove that $\langle w_n \rangle$ is Cauchy as well.

6. The graph of a function $f : X \to Y$ is the set

$$\Gamma(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y .$$

Suppose X, Y are topological spaces and $f : X \to Y$ is a continuous function. Prove that $\Gamma(f)$ is path connected, if X is path connected.

Recall that a topological space S is path connected, if for every $x, y \in S$ there exists a continuous function $p : [0,1] \to S$ of the unit interval into S, such that, $p(0) = x, p(1) = y$.

7. (a) Suppose X, Y are topological spaces. Define what it means to say that a function $f : X \to Y$ is continuous.

(b) Use your definition above to show that, for every sequence $\langle x_n \rangle \subseteq X$ converging to $x \in X$, the sequence $\langle f(x_n) \rangle$ converges to $f(x)$.

(c) By using your answers above, determine if the following real valued function of the real numbers (equipped with the usual topology), is continuous:

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin(1/x) & \text{if } x \neq 0. \end{cases}$$