Fall 2014 Topology Comprehensive Exam
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Do any five (5) of the problems that follow. Each is worth 20 points. Please indicate clearly which five you want us to grade. (Otherwise, we will grade your first five answers.)

1. Let $\mathbb{N}$ denote the set of positive integers. Let $\mathcal{T}$ be the family consisting of $\emptyset, \mathbb{N}$, and all sets of the form $\{1, 2, 3, \ldots, n\}$.
    a. Prove that $\mathcal{T}$ is a topology for $\mathbb{N}$.
    b. Find the closure of $\{1\}$ in this topology, and prove that your answer is correct.

2. Recall that if X is a topological space and A is a subset of X and $p \in X$, then we say that p is a *limit point* of A if every open neighborhood of p contains a point of A other than p.
    a. Let X be a Hausdorff space, and let S be a compact subset of X. Let L be the set of all limit points of S. Prove that L is compact.
    b. Give an example to show that (a) may fail when X is not Hausdorff.

3. Show a topological space X is connected if and only if every continuous function from X to a discrete space is constant.

4. Suppose $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ are connected topological spaces. Prove that $X \times Y$ equipped with the product topology is connected.

5. Let $\langle X, d \rangle$ be a metric space.
    (a) Prove that for $p \in X$ and $\alpha > 0$, the set $S_d(p, \alpha) := \{x \in X \mid d(p, x) < \alpha\}$ is open.
    (b) Prove that each Cauchy sequence $\langle x_n \rangle$ in X is bounded, that is, $\{x_n \mid n \in \mathbb{N}\}$ is contained in some ball $S_d(p, \alpha)$.

6. Let $f, f_1, f_2, f_3, \ldots$ be real-valued continuous functions defined on a topological space $\langle X, \tau \rangle$.
    (a) What does it mean for $\langle f_n \rangle$ to converge uniformly to f?
    (b) Suppose X is compact and that $f_{n+1}(x) \leq f_n(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$. Suppose that for all $x \in X$, we have that

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Prove that $\langle f_n \rangle$ converges uniformly to f. Hint: Let $g_n = f_n - f$.

7. Let Y be a nonempty set. Endow Y with the co-countable topology, so that a subset A of Y is closed if and only if $A = Y$ or A is countable. Prove that Y is Lindelöf. (Recall that we say a topological space X is *Lindelöf* if every open cover of X has a countable subcover.)