Fall 2017 Topology Comprehensive Exam

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Do any five (5) of the problems that follow. Each is worth 20 points. Please indicate clearly which five you want us to grade. (Otherwise, we will grade your first five answers.)

(1) Prove or disprove each statement in the context of metric spaces.
   (a) The finite union of a family $\{B_1, B_2, \ldots, B_n\}$ of nonempty bounded subsets of (X, d) is bounded.
   (b) If $f : (X, d) \to (Y, \rho)$ is continuous and $B \subseteq X$ is bounded, then $f(B)$ is bounded.

(2) Let (X, τ) be a connected topological space and let $f : X \to \mathbb{R}$ be continuous. Equip $\mathbb{R}$ with the usual topology, and equip $X \times \mathbb{R}$ with the product topology.
   (a) Prove that $\{(x, f(x)) : x \in X\}$ is a connected subset of $X \times \mathbb{R}$.
   (b) Let $p \in X$. Prove that $\{(p, y) : y \geq f(p)\}$ is a connected subset of $X \times \mathbb{R}$. [Hint: You may assume without proof that in $\mathbb{R}$, all closed rays are connected.]
   (c) Prove that $\{(x, y) : x \in X, y \geq f(x)\}$ is a connected subset of $X \times \mathbb{R}$. [Hint: Use (a) and (b).]

(3) Let (X, τ) be a topological space.
   (a) Explain what *second countability, first countability* and *separability* of (X, τ) all mean.
   (b) Prove that each second countable space is separable.
   (c) Give an example of a first countable space that is not separable.

(4) Let $\langle x_n \rangle$ be a sequence in a topological space (X, τ), not assumed convergent.
   (a) What does it mean for $p \in X$ to be a *cluster point* of $\langle x_n \rangle$?
   (b) Produce a sequence with distinct terms in $\mathbb{R}$ equipped with the usual topology whose set of cluster points is $\{1, 5, 7\}$.
   (c) Produce a sequence with distinct terms in $\mathbb{R}$ equipped with the usual topology whose set of cluster points is $\mathbb{R}$ itself.

(5) Let X be a topological space. Prove that if every countable subset of X is compact, then X is compact. [Hint: Do a proof by contradiction. Use the definition of "compact."]

(6) (a) Define what it means for a topological space to be *path connected*.
   (b) Give an example of a compact and connected topological space that is not path connected.

(7) Let $\{(x, y) : x^2 + y^2 \leq 1\}$ be the unit closed disk in the plane $\mathbb{R}^2$. Prove or disprove that the punctured open disk $\{(x, y) : 0 < x^2 + y^2 < 1\}$ is homeomorphic to $\mathbb{R}^2 \setminus D$.