# Fall 2018 Topology Comprehensive Exam

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Do any five (5) of the problems that follow. Each is worth 20 points. Please indicate clearly which five you want us to grade. (Otherwise, we will grade your first five answers.)

(1) Let $(X, d)$ be a metric space. Let $\{x_n\}$ be a Cauchy sequence in $X$. Prove the following statements.
   (a) If $\{x_{n_i}\}$ is a subsequence that converges to $x$, then $\{x_n\}$ converges to $x$.
   (b) If $\{w_n\}$ is a sequence such that $d(x_n, w_n) \to 0$, then $\{w_n\}$ is a Cauchy sequence.
   (c) The set $\{x_n \mid n \in \mathbb{N}\}$ is bounded.

(2) (a) Show that if $X_1, \ldots, X_n$ are discrete topological spaces, then the product space $X_1 \times \cdots \times X_n$ is discrete.
   (b) Give a counterexample to show that if $\{X_k\}_{k=1}^{\infty}$ is an infinite collection of discrete topological spaces, then the product space $\Pi_{k=1}^{\infty} X_k$ may not be discrete. Hint: For all $k$, let $X_k = \{0, 1\}$ with the discrete topology.

(3) In this problem we take the usual (that is, Euclidean) topology on $\mathbb{R}^2$ and $\mathbb{R}^3$.
   (a) Is $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 \neq 1\}$ a connected subspace of $\mathbb{R}^2$? Prove that your answer is correct.
   (b) Is $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$ a connected subspace of $\mathbb{R}^3$? Prove that your answer is correct.

(4) Let $X$ and $Y$ be topological spaces such that $Y$ is Hausdorff. Let $f$ and $g$ be continuous functions from $X$ to $Y$. Show that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of $X$.

(5) Let $X$ and $Y$ be topological spaces. Let $f\colon X \to Y$ be continuous. Let $A \subseteq X$. Give a counterexample to show that the statement $\overline{f(A)} = f(\overline{A})$ is not always true.

(6) (a) Let $X$ and $Y$ be topological spaces such that $X$ is compact and $Y$ is Hausdorff. Prove that if $f\colon X \to Y$ is continuous and bijective, then $f$ is a homeomorphism.
   (b) Let $X$ be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on $X$ such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Show that if $(X, \mathcal{T}_1)$ is Hausdorff and $(X, \mathcal{T}_2)$ is compact, then $\mathcal{T}_1 = \mathcal{T}_2$. Hint: Use (a).

(7) For a topological space $X$, and subsets $A$ and $B$ of $X$, indicate which of the following statements are true. In each case, justify your answer by providing a proof of the true statements and a counterexample for the false statements.
   (a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
   (b) $\overline{A \cap B} = \overline{A} \cap \overline{B}$