Comprehensive Examination – Topology

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Do any five of the problems that follow. Each problem is worth 20 points. The set of positive integers is denoted by N, the set of rationals by Q, and the set of real numbers by R. The notation A^c means the complement of the set A with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Let (X, τ) and (Y, σ) be topological spaces and let $f : X \to Y$ be continuous.

(a) Let A be a connected subset of X. Prove that $f(A)$ is a connected subset of Y.

(b) Give an example showing that "connected" cannot be replaced by "closed".

2. Let (X, d) be a metric space.

(a) Let $x_0 \in X$ and let $r > 0$. Prove that the closed ball $B[x_0, r] = \{x \in X : d(x, x_0) \leq r\}$ is a closed subset in (X, d).

(b) Prove that X is regular.

3. Let (X, τ) and (Y, σ) be topological spaces and let $f : X \to Y$ be continuous.

(a) Assuming that Y is Hausdorff, prove that the graph of f, $\Gamma(f) = \{(x, y) : x \in X, y = f(x)\}$ is a closed subset of $X \times Y$ equipped with the product topology.

(b) Prove that $\Gamma(f)$ equipped with the relative topology is homeomorphic to X.

4. Suppose that the topological space (X, τ) has a countable base.

(a) Show that if $\{V_\alpha : \alpha \in \Delta\}$ is a family of open, pairwise disjoint, nonempty subsets of X, then Δ must be countable.

(b) Let A be an uncountable subset of X. Prove that some point of X must be a limit point of A. (Hint: if not, consider A as a subset of X.)

5. Let (X, τ) be a Hausdorff space. We say that a sequence $(x_n)_{n=1}^{\infty}$ is *convergent to* $x \in X$ iff for each neighborhood V of x there exists $n \in N$ such that for each $k > n$ we have $x_k \in V$. In this case we write $(x_n) \to x$.

(a) Suppose that $(x_n) \to x$ and $(x_n) \to y$. Prove that $x = y$.

(b) Suppose that $(x_n) \to x$. Prove that $\{x_n : n \in N\} \cup \{x\}$ is a compact subset of X.

6. The *lower limit topology* on R, a.k.a. the *Sorgenfrey topology*, is the topology τ_L having as a base all half-open intervals $[a, b)$ where $a < b$.

(a) Is the space (R, τ_L) first countable? Explain.

(b) Is the space (R, τ_L) connected? Explain.

(c) Is $[0, 1]$ compact as a subspace of (R, τ_L)? Explain.

7. Let (X, τ) be a topological space and (Y, σ) be compact topological space. Suppose that F is a closed subset of $X \times Y$ and π_1 is the usual projection map from $X \times Y$ to X. Show that if $x_0 \in X \setminus \pi_1(F)$, then there exists a neighborhood U of x_0 such that $F \cap (U \times Y) = \emptyset$.