Spring 2009 Topology Comprehensive Exam DRAFT 5-18-2009

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Do any five (5) of the problems that follow. Each is worth 20 points.

1. Two metrics d and d' on a set X are called *metrically equivalent* if the identity map $i : (X, d) \to (X, d')$ and its inverse are both uniformly continuous. Show that if d and d' are metrically equivalent then X is complete under d if and only if X is complete under d'.

2. Consider the topology τ on $\mathbb{R}$ having as a base all sets of the form $[a, b)$ where $a < b$.
(a) Show τ is properly finer than the usual topology τ_u.
(b) Is $(\mathbb{R}, \tau)$ connected? Justify your answer.

3. Let X be a topological space. Show that the following two assertions are equivalent:
(a) Any subset of X is normal.
(b) For every pair A, B of subsets of X such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, there exist disjoint open sets containing them.

4. (a) Prove that each second countable space is first countable.

(b) Prove that each separable metric space is second countable.

5. Let $\mathbb{R}$ denote the set of real numbers. Let $X = \{(a, b) \mid a, b \in \mathbb{R}^2\} \setminus \{(0, 0)\}$. (In other words, X equals the plane minus the origin.) Let

$$\tau = \{X \setminus \ell \mid \ell \text{ is a line through the origin in } \mathbb{R}^2\},$$

and endow X with the topology generated by τ.

(a) Prove that every open set in X equals $\emptyset$, X, or a set of the form $X \setminus (\ell_1 \cup \cdots \cup \ell_n)$ for some positive integer n and lines $\ell_1, \ldots, \ell_n$ through the origin.
(b) Prove that X is compact.

6. Prove that X is Hausdorff iff $\{(x, x) \mid x \in X\}$ is closed in $X \times X$ equipped with the product topology.

7. (a) Show that $f(x) = \dfrac{x}{1 - |x|}$ defines a homeomorphism of the open interval $(-1, 1)$ onto $\mathbb{R}$. (Here $\mathbb{R}$ denotes the set of real numbers in the usual topology.)
(b) Give an example of a homeomorphism of $(2, 7)$ onto $\mathbb{R}$.