Fall 2010 Topology comprehensive exam

Krebs* and Beer

Please do any FIVE of the seven problems below. They are worth 20 points each. Indicate CLEARLY which five you want us to grade—if you do more than five problems, we will select five to grade, and they may not be the five that you want us to grade.

Let $\mathbb{R}$ denote the set of real numbers. Unless otherwise stated, assume that $\mathbb{R}^n$ is endowed with the usual topology, and that subspaces of $\mathbb{R}^n$ are endowed with the subspace topology.

(1) Let $\mathscr{S}_1$ be the collection of all sets of the form $(-\infty, \alpha) \times (-\infty, \beta)$, where $\alpha, \beta \in \mathbb{R}$. Let $\mathscr{S}_2$ be the collection of all sets of the form all sets of the form $(\lambda, \infty) \times (\mu, \infty)$, where $\lambda, \mu \in \mathbb{R}$. Let $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2$. Show that $\mathscr{S}$ is a subbasis for the topology on $\mathbb{R}^2$ determined by the Euclidean metric $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

(2) Prove that $\mathbb{R}$ is Lindelöf. (Recall that we say a topological space X is *Lindelöf* if every open cover of X has a countable subcover.)

(3) Let A and B be connected subspaces of a topological space X with a common point p. Prove or give a counterexample to each statement.
   (a) $A \cap B$ is connected;
   (b) $A \cup B$ is connected.

(4) Let $A = \{(x, y, 0) \ : \ x^2 + y^2 = 1\} \subset \mathbb{R}^3$. Let $B = \{(0, y, z) \ : \ y^2 + (z-1)^2 = 1\} \subset \mathbb{R}^3$. Let $X = A \cup B$. What are the path components of X? Prove that your answer is correct.

(5) Let X be a compact Hausdorff space. Prove that a subspace A is compact in the subspace topology if and only if A is closed in X.

(6) In a metric space $\langle X, d \rangle$, let $B_d(x, \alpha)$ denote the open ball with center $x \in X$ and radius $\alpha > 0$. Call a subset A *bounded* if A is contained in some open ball, and *totally bounded* if $\forall \varepsilon > 0$, $\exists$ a finite subset F of X such that $A \subseteq \cup_{x \in F} B_d(x, \varepsilon)$.
   (a) Prove that if $A \subseteq X$ is totally bounded, then A is bounded;
   (b) Prove that if $A \subseteq X$ is compact, then A is totally bounded.

(7) Let $X = \mathbb{R}^2$ with the usual topology. Let $Y = \mathbb{R}$ with the cofinite topology. (That is, A is closed in Y iff A is finite or A is Y.) Define $f : X \to Y$ by $f(a, b) = a + b$. Prove that f is continuous.