TOPOLOGY COMPREHENSIVE EXAMINATION SPRING 2014

(Akis*, Beer, Krebs)

DO 5 OUT OF 7 (Indicate clearly which 5 problems you submit for evaluation)

1. Let I be an infinite set. For each $\alpha \in I$, let $X_\alpha = \{0,1\}$ in the discrete topology. For every S subset of I, define

$$f_S = \langle x_\alpha \rangle \in \prod_{\alpha \in I} X_\alpha, \text{ where } x_\alpha = 1 \text{ if } \alpha \in S, \text{ and } x_\alpha = 0 \text{ if } \alpha \notin S.$$

Let $A = \{f_S : S \text{ is finite}\}$. Show that $f_I \in \overline{A}$, where $\overline{A}$ is the closure of A in the product space $\prod_{\alpha \in I} X_\alpha$.

2. Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on a set X. Prove that $\{U \cap V : U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$ is a basis for the coarsest topology on X containing $\mathcal{T}_1 \cup \mathcal{T}_2$.

3. Let X be a topological space. Let A, B, and C be connected subsets of X such that $A \cap B = A \cap C = \emptyset$.

(a) Recall that we say a set is "clopen" if it is both open and closed. Prove that if U is a proper nonempty clopen subset of $A \cup B$, then $U = A$ or $U = B$.

(b) Suppose that $A \cup B$ is disconnected. Prove that if $A \cup B$ is homeomorphic to $A \cup C$, then B is homeomorphic to C. Hint: This is the second part of a two-part question.

4. (a) Prove that each metric space (X, d) is first countable.

(b) Give an example of a topological space $(X, \mathcal{T})$ that is not first countable. Prove that your answer is correct.

5. (a) Let A, B be subsets of a topological space $(X, \mathcal{T})$. Prove $\overline{A \cup B} = \overline{A} \cup \overline{B}$, i.e. the closure of the union of A and B equals the union of their closures.

(b) Suppose A and B have compact closures. Prove that $\overline{A \cup B}$ is also compact.

6. (a) Prove the continuous image of a compact topological space is compact.

(b) Suppose $(X, \mathcal{T})$ is a compact and connected topological space. Show that the image of every non-constant, continuous, real valued function $f: X \to R$, is a closed interval.

7. We call p a *cluster point* of a sequence $\langle x_n \rangle$ in a metric space (X,d), if for all $\varepsilon > 0$ and for every natural number k, there exists $n > k$ with $d(x_n, p) < \varepsilon$.

(a) Prove that the set of cluster points of a sequence $\langle x_n \rangle$ in a metric space (X,d), is closed.

(b) Give an example of a sequence of real numbers whose cluster point set is the closed interval $[0,1]$.